Why A Positive Semi-Definite Matrix is Positive Definite if its Determinant is Not Zero?

Here we show the following:

> Suppose a matrix $\mathbf{A}$ is symmetric and positive semi-definite (in that $\mathbf{x}'\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x}$). Then it is positive definite if $|\mathbf{A}| \neq 0$.

Perhaps the easiest way to prove is to use the following well-known results about eigenvalues.

1. The eigenvalues of a symmetric matrix are all non-negative if and only if the matrix is positive semi-definite (in that $\mathbf{x}'\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x}$).

2. The eigenvalues of a symmetric matrix are all positive if and only if the matrix is positive definite.

3. The product of the eigenvalues of a matrix equals its determinant.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of an n-dimensional symmetric positive semi-definite matrix $\mathbf{A}$. Then by the first result, $\lambda_i \geq 0$ for all i. By the third result, $\lambda_1 \times \cdots \times \lambda_n = |\mathbf{A}|$. Thus if the determinant of a symmetric positive semi-definite matrix is not zero, then the eigenvalues are all positive. By the second result, such a matrix is positive definite.