Incorporating Serial Correlation in Multiple Equations

Having developed techniques for incorporating serial correlation for multiple-equation GMM, doing the same for multiple-equation GMM is straightforward, since multiple-equation GMM is a special case of single-equation GMM. We use "t" for the observation index here, because the issue at hand is serial correlation between observations.

Estimation of S Without Conditional Homoskedasticity

Define $\mathbf{g}_t$ as in (4.1.4):

$$\underset{(\sum_{m=1}^{M} K_m \times 1)}{\mathbf{g}_t} \equiv \begin{bmatrix} \mathbf{x}_{t1} \cdot \varepsilon_{t1} \\ \vdots \\ \mathbf{x}_{tM} \cdot \varepsilon_{tM} \end{bmatrix}. \tag{1}$$

Given this $\mathbf{g}_t$, the definition of $\mathbf{S}$ for multiple-equations is the same as in (6.6.1) and (6.6.2). The expression for $\mathbf{\Gamma}_j$ explicitly recognizing multiple equations is

$$\begin{aligned} \mathbf{\Gamma}_j = & \underset{(\sum_{m=1}^{M} K_m \times \sum_{m=1}^{M} K_m)}{\mathrm{E}(\mathbf{g}_t \mathbf{g}'_{t-j})} \\ = & \begin{bmatrix} \mathrm{E}(\varepsilon_{t1}\varepsilon_{t-j,1}\mathbf{x}_{t1}\mathbf{x}'_{t-j,1}) & \cdots & \mathrm{E}(\varepsilon_{t1}\varepsilon_{t-j,M}\mathbf{x}_{t1}\mathbf{x}'_{t-j,M}) \\ \vdots & & \vdots \\ \mathrm{E}(\varepsilon_{tM}\varepsilon_{t-j,1}\mathbf{x}_{tM}\mathbf{x}'_{t-j,1}) & \cdots & \mathrm{E}(\varepsilon_{tM}\varepsilon_{t-j,M}\mathbf{x}_{tM}\mathbf{x}'_{t-j,M}) \end{bmatrix}. \end{aligned} \tag{2}$$

To estimate $\mathbf{S}$ by the kernel-based method, we need to estimate $\widehat{\mathbf{\Gamma}}_j$ ($j = 0, 1, 2, \dots$). But the estimation is the same as in (6.6.3) on p. 408, with $\widehat{\mathbf{g}}_t$ defined as

$$\widehat{\mathbf{g}}_t \equiv \begin{bmatrix} \mathbf{x}_{t1} \cdot \widehat{\varepsilon}_{t1} \\ \vdots \\ \mathbf{x}_{tM} \cdot \widehat{\varepsilon}_{tM} \end{bmatrix}. \tag{3}$$

Given $\{\widehat{\mathbf{\Gamma}}_j\}$, the kernel-based estimation of $\mathbf{S}$ proceeds exactly as described on pp. 408-410. Given $\{\widehat{\mathbf{g}}_t\}$, the VARHAC estimation of $\mathbf{S}$ is exactly as described on pp. 410-412.

Estimation of S under Conditional Homoskedasticity

Estimation of $\mathbf{S}$ under conditional homoskedasticity for single equations is described in Section 6.7. It should be clear how the discussion can be adapted to multiple equations. The conditional homoskedasticity assumption is

$$\mathrm{E}(\varepsilon_{mt}\varepsilon_{h,t-j} | \mathbf{x}_{mt}, \mathbf{x}_{h,t-j}) = \mathrm{E}(\varepsilon_{mt}\varepsilon_{h,t-j}) = \omega_{mhj} \quad (m, h = 1, 2, \dots, M). \tag{4}$$

For later use, define

$$\underset{(M \times 1)}{\boldsymbol{\varepsilon}_t} \equiv \begin{bmatrix} \varepsilon_{1t} \\ \vdots \\ \varepsilon_{Mt} \end{bmatrix}, \tag{5}$$

and let $\underset{(M\times M)}{\boldsymbol{\Omega}_j}$ be the j-th autocovariance matrix of $\{\boldsymbol{\varepsilon}_t\}$:

$$\underset{(M\times M)}{\boldsymbol{\Omega}_j} \equiv \mathrm{E}(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_{t-j}) = \begin{bmatrix} \mathrm{E}(\varepsilon_{1t}\varepsilon_{1,t-j}) & \cdots & \mathrm{E}(\varepsilon_{1t}\varepsilon_{M,t-j}) \\ \vdots & & \vdots \\ \mathrm{E}(\varepsilon_{Mt}\varepsilon_{1,t-j}) & \cdots & \mathrm{E}(\varepsilon_{Mt}\varepsilon_{M,t-j}) \end{bmatrix} = (\omega_{mhj}). \tag{6}$$

The usual argument utilizing the Law of Total Expectation (used in (6.7.3) for single equations) implies that

$$\underset{(\sum_{m=1}^M K_m \times \sum_{m=1}^M K_m)}{\boldsymbol{\Gamma}_j} = \begin{bmatrix} \omega_{11j}\,\mathrm{E}(\mathbf{x}_{t1}\mathbf{x}'_{t-j,1}) & \cdots & \omega_{1Mj}\,\mathrm{E}(\mathbf{x}_{t1}\mathbf{x}'_{t-j,M}) \\ \vdots & & \vdots \\ \omega_{M1j}\,\mathrm{E}(\mathbf{x}_{tM}\mathbf{x}'_{t-j,1}) & \cdots & \omega_{MMj}\,\mathrm{E}(\mathbf{x}_{tM}\mathbf{x}'_{t-j,M}) \end{bmatrix}. \tag{7}$$

Estimation of $\boldsymbol{\Gamma}_j$ $(j = 0, 1, \dots)$ exploits this structure, by estimating $\boldsymbol{\Omega}_j$ $(= (\omega_{mhj}))$ and $\mathrm{E}(\mathbf{x}_{tm}\mathbf{x}'_{t-j,h})$ separately. Let $\widehat{\boldsymbol{\varepsilon}}_t$ be the vector of estimated errors. $\boldsymbol{\Omega}_j$ can be estimated as

$$\underset{(M\times M)}{\widehat{\boldsymbol{\Omega}}_j} = \frac{1}{n}\sum_{t=j+1}^{n} \widehat{\boldsymbol{\varepsilon}}_t\widehat{\boldsymbol{\varepsilon}}'_{t-j}. \tag{8}$$

Similarly, $\mathrm{E}(\mathbf{x}_{tm}\mathbf{x}'_{t-j,h})$ can be estimated in the obvious way:

$$\underset{(K_m\times K_h)}{\mathrm{E}(\widehat{\mathbf{x}_{tm}\mathbf{x}'_{t-j,h}})} = \frac{1}{n}\sum_{t=j+1}^{n} \mathbf{x}_{tm}\mathbf{x}'_{t-j,h}. \tag{9}$$

Therefore, if $\widehat{\omega}_{mhj}$ is the (m,h) element of $\widehat{\boldsymbol{\Omega}}_j$, our estimate of $\boldsymbol{\Gamma}_j$ is

$$\widehat{\boldsymbol{\Gamma}}_j = \begin{bmatrix} \widehat{\omega}_{11j}\,\frac{1}{n}\sum_{t=j+1}^{n}\mathbf{x}_{t1}\mathbf{x}'_{t-j,1} & \cdots & \widehat{\omega}_{1Mj}\,\frac{1}{n}\sum_{t=j+1}^{n}\mathbf{x}_{t1}\mathbf{x}'_{t-j,M} \\ \vdots & & \vdots \\ \widehat{\omega}_{M1j}\,\frac{1}{n}\sum_{t=j+1}^{n}\mathbf{x}_{tM}\mathbf{x}'_{t-j,1} & \cdots & \widehat{\omega}_{MMj}\,\frac{1}{n}\sum_{t=j+1}^{n}\mathbf{x}_{tM}\mathbf{x}'_{t-j,M} \end{bmatrix} \tag{10}$$

for $j = 0, 1, 2, \dots$. This is the multiple-equation version of (6.7.4) on p. 414. Given these estimated autocovariances, the kernel-based estimation of $\widehat{\mathbf{S}}$ is exactly the same is in the case without conditional homoskedasticity discussed above.