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Hayashi *Econometrics*: Answers to Selected Review Questions

Chapter 9

Section 9.1

1. By the hint, the long-run variance equals $\mathrm{Var}((u_T - u_0)/\sqrt{T}) = \frac{1}{T}\mathrm{Var}(u_T - u_0)$. $\mathrm{Var}(u_T - u_0) = \mathrm{Var}(u_T) + \mathrm{Var}(u_0) - 2\rho(u_T, u_0)\sqrt{\mathrm{Var}(u_T)}\sqrt{\mathrm{Var}(u_0)}$. Since the correlation coefficient $\rho(u_T, u_0)$ is less than or equal to 1 in absolute value and since $\mathrm{Var}(u_T)$ and $\mathrm{Var}(u_0)$ are finite, $\mathrm{Var}(u_T - u_0$ is finite.

Section 9.2

3. $\alpha_0 = 1$, $\alpha_1 = -1$, and $\alpha_j = 0$ for $j = 2, 3, ...$ So $\eta_t = \varepsilon_t - \varepsilon_{t-1}$.

Section 9.3

1. $T^{1-\eta}(\widehat{\rho} - 1) = \frac{1}{T^\eta}T(\widehat{\rho} - 1)$. $T(\widehat{\rho} - 1)$ converges in distribution to a random variable. Use Lemma 2.4(b).

2. This follows immediately from Proposition 9.2(a),(b), and (9.3.3).

3. Since Δy_t is ergodic stationary (actually, iid here), $\frac{1}{T-1}\sum_{t=1}^{T}(\Delta y_t)^2 \to_{\mathrm{p}} \mathrm{E}(\Delta y_t)$. By Proposition 9.3, $[T \cdot (\widehat{\rho} - 1)]$ converges in distribution to a random variable, and by Proposition 9.2(b), $\frac{1}{T}\sum_{t=1}^{T}\Delta y_t\, y_{t-1}$ converges in distribution to a random variable. So the second term converges in probability to zero. Use a similar argument to show that the third term vanishes.

4. Δy_t is stationary, so for the t-value from the first regression you should use the standard normal. The t-value from the second regression is numerically equal to (9.3.7). So use DF_t.

5. (a) As remarked on page 564, an I(0) process is ergodic stationary. So by the ergodic theorem

$$\widehat{\rho} = \frac{\frac{1}{T}\sum_{t=1}^{T} y_t y_{t-1}}{\frac{1}{T}\sum_{t=1}^{T} y_t^2} \underset{\mathrm{p}}{\to} \frac{\gamma_1}{\gamma_0},$$

where $\gamma_0 = \mathrm{E}(y_t^2)$ and $\gamma_1 = \mathrm{E}(y_t y_{t-1})$. By assumption, $\gamma_0 > \gamma_1$.

(b) It should be easy to show that

$$s^2 \underset{\mathrm{p}}{\to} \frac{2(\gamma_0^2 - \gamma_1^2)}{\gamma_0} > 0.$$

So

$$\sqrt{T}\cdot t = \frac{\widehat{\rho} - 1}{s \div \sqrt{\frac{1}{T}\sum_{t=1}^{T}(y_{t-1})^2}} = -\frac{\gamma_0 - \gamma_1}{\sqrt{2(\gamma_0 - \gamma_1)}} < 0.$$

7. (a) SB times T is the reciprocal of DW with y_t interpreted as the regression residual.

(b) The denominator of SB converges in distribution to $\mathrm{E}[(\Delta y_t)^2] = \gamma_0$. By Proposition 9.2(a), the numerator converges in distribution to $\lambda^2 \int_0^1 [W(r)]^2 \, \mathrm{d}r$. Here, $\lambda^2 = \gamma_0$.

(c) If y_t is I(0),

$$T \cdot SB \underset{\mathrm{p}}{\rightarrow} \frac{\mathrm{E}(y_t^2)}{\mathrm{E}[(\Delta y_t)^2]}.$$

Section 9.4

1. $a_1 = \phi_1 + \phi_2 + \phi_3$, $a_2 = -\phi_2$, $a_3 = -\phi_3$. If y_t is driftless I(1) following (9.4.1), then y_{t-1} is driftless I(1) while $y_{t-1} - y_{t-2}$ and $y_{t-1} - y_{t-3}$ is zero-mean I(0). $\mathbf{a} \equiv (a_1, a_2, a_3)'$ is a linear and non-singular transformation of $\boldsymbol{\phi} \equiv (\phi_1, \phi_2, \phi_3)'$ (that is, $\mathbf{a} = \mathbf{F}\boldsymbol{\phi}$ for some non-singular matrix $\mathbf{F}$). So if $\widehat{\boldsymbol{\phi}}$ is the OLS estimate of $\boldsymbol{\phi}$, then $\mathbf{F}\widehat{\boldsymbol{\phi}}$ is the OLS estimate of $\mathbf{a}$. (ρ, ζ_1, ζ_2) from (9.4.3) with $p = 2$ is also a linear and non-singular transformation of $\boldsymbol{\phi}$. $\rho = a_1 = \phi_1 + \phi_2 + \phi_3$.

2. Just apply the mean value theorem to $\phi(z)$.

3. The hint is almost the answer. In the final step, use the fact that $\frac{1}{T}\sum_{t=1}^{T}(\Delta y_t)^2 \rightarrow_{\mathrm{p}} \gamma_0$.

4. (a) The hint is the answer. (b) Use Billingsley's CLT. Δy_{t-1} is a function of $(\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots)$. So Δy_{t-1} and ε_t are independently distributed, and $\mathrm{E}[(\Delta y_{t-1}\, \varepsilon_t)^2] = \mathrm{E}[(\Delta y_{t-1})^2]\, \mathrm{E}(\varepsilon_t^2) = \gamma_0 \sigma^2$.

5. The hint is the answer.

6. The hint is almost the answer. We have shown in Review Question 3 to Section 9.3 that $s^2 \rightarrow_{\mathrm{p}} \sigma^2$. It has been shown on page 588 that the (2,2) element of $\mathbf{A}_T^{-1}$ converges in probability to γ_0^{-1}.