Proof that $\widetilde{\mathbf{X}}_2'\widetilde{\mathbf{X}}_2$ on p. 73 is invertible

The proof is as follows. It suffices to show that $\widetilde{\mathbf{X}}_2$ $(n \times K_2)$ is of full column rank. Suppose not. Then there exists a K_2-dimensional non-zero vector $\boldsymbol{\alpha}$ such that $\widetilde{\mathbf{X}}_2\boldsymbol{\alpha} = \mathbf{0}$. Since $\widetilde{\mathbf{X}}_2 = \mathbf{M}_1\mathbf{X}_2$, we have:

$$
\begin{aligned}
\mathbf{0} = \widetilde{\mathbf{X}}_2\boldsymbol{\alpha} &= \mathbf{X}_2\boldsymbol{\alpha} - \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2\boldsymbol{\alpha} \\
&= \mathbf{X}_2\boldsymbol{\alpha} + \mathbf{X}_1\boldsymbol{\gamma} \quad (\text{where } \boldsymbol{\gamma} \equiv -(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2\boldsymbol{\alpha}) \\
&= \mathbf{X}\boldsymbol{\pi} \quad (\text{where } \boldsymbol{\pi} \equiv \begin{bmatrix} \boldsymbol{\gamma} \\ \boldsymbol{\alpha} \end{bmatrix}).
\end{aligned}
$$

Since $\mathbf{X}$ is of full column rank and since $\boldsymbol{\pi}$ is a non-zero vector, this is a contradiction.