Proof of (7.2.14) on p. 466

Since $0 < \Phi(v) < 1$, $\log \Phi(v) < 0$ for all v. So (7.2.14) can be restated as

$$g(v) \equiv -[\log \Phi(v) - \log \Phi(0)] \leq |v| + |v|^2 \qquad (*)$$

for all v. We have: $g(0) = 0$ and $g'(v) = -\frac{\phi(v)}{\Phi(v)} < 0$ (where $\phi(.)$, the first derivative of $\Phi(v)$, is the standard normal density function). Thus $g(v) < 0$ for all $v > 0$ and $(*)$ holds for $v > 0$. In what follows, we prove $(*)$ for $v \leq 0$. To avoid confusion, define $x \equiv -v$. Then $x \geq 0$ and $(*)$ can be rewritten as

$$h(x) \equiv -[\log \Phi(-x) - \log \Phi(0)] \leq x^2 + x. \qquad (**)$$

Noting that $\Phi(-x) = 1 - \Phi(x)$ and $\phi(-x) = \phi(x)$, we observe

$$h'(x) = \frac{\phi(-x)}{\Phi(-x)} = \frac{\phi(x)}{1 - \Phi(x)} \equiv \lambda(x).$$

This ratio $\lambda(x)$ is called the **inverse Mill's ratio** (as noted on p. 478). It is well known (and can be verified by taking the derivative of $\lambda(x)$) that $\lambda(x)$ is monotonically increasing, convex and asymptotes to x. Its value at $x = 0$ equals $2\phi(0) = \sqrt{\frac{2}{\pi}}$, which is about 0.80. On the other hand, the derivative of $x^2 + x$ has a slope of 2 and its value at $x = 0$ is 1. So the slope of the RHS of $(**)$ is strictly steeper than the slope of the LHS ($\lambda(x)$) for all $x \geq 0$. Since the RHS at $x = 0$ equals the LHS at $x = 0$, this means that the LHS is less than the RHS for all $x > 0$.