Proof of (1.2.19) on p. 21

Two proofs are given here. The second proof, which is more straightforward with brute force, was provided by K. Fukushima of University of Tokyo.

Proof 1.

Let $\mathbf{y}^{(i)}$ be an $n \times 1$ vector obtained by replacing the i-th element of $\mathbf{y}$ by zero. Define $\mathbf{X}^{(i)}$ similarly. Let $\mathbf{v}^{(i)}$ be an n-dimensional vector whose i-th element is 1 and whose other elements are all zero. Thus

$$\underset{(n\times 1)}{\mathbf{y}^{(i)}} \equiv \begin{bmatrix} y_1 \\ \vdots \\ y_{i-1} \\ 0 \\ y_{i+1} \\ \vdots \\ y_n \end{bmatrix}, \qquad \underset{(n\times K)}{\mathbf{X}^{(i)}} \equiv \begin{bmatrix} \mathbf{x}_1' \\ \vdots \\ \mathbf{x}_{i-1}' \\ \mathbf{0}' \\ \mathbf{x}_{i+1}' \\ \vdots \\ \mathbf{x}_n' \end{bmatrix}, \qquad \underset{(n\times 1)}{\mathbf{v}^{(i)}} \equiv \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{1}$$

Thus, for example, $\mathbf{y} - \mathbf{y}^{(i)} = y_i\,\mathbf{v}^{(i)}$, $\mathbf{X} - \mathbf{X}^{(i)} = \mathbf{v}^{(i)}\mathbf{x}_i{}'$, and $\mathbf{X}'\mathbf{v}^{(i)} = \mathbf{x}_i$.

Clearly, $\mathbf{b}^{(i)}$, the OLS estimator obtained by dropping the i-th observation, equals the OLS estimator from regressing $\mathbf{y}^{(i)}$ on $\mathbf{X}^{(i)}$. Define

$$\underset{(n\times 1)}{\mathbf{e}^{(i)}} \equiv \mathbf{y}^{(i)} - \mathbf{X}^{(i)}\mathbf{b}^{(i)}. \tag{2}$$

By construction, the i-th element of $\mathbf{e}^{(i)}$ is zero. It should be easy to show:

$$\mathbf{v}^{(i)\prime}\mathbf{e}^{(i)} = 0, \tag{a}$$

$$\mathbf{X}'\mathbf{e}^{(i)} = \mathbf{X}^{(i)\prime}\mathbf{e}^{(i)} = \mathbf{0}. \tag{b}$$

From (2) and the definition of $\mathbf{e}$ (that $\mathbf{e} \equiv \mathbf{y} - \mathbf{X}\mathbf{b}$), derive

$$\begin{aligned}
\mathbf{e} - \mathbf{e}^{(i)} &= (\mathbf{y} - \mathbf{X}\mathbf{b}) - (\mathbf{y}^{(i)} - \mathbf{X}^{(i)}\mathbf{b}^{(i)}) \\
&= (\mathbf{y} - \mathbf{y}^{(i)}) - (\mathbf{X}\mathbf{b} - \mathbf{X}^{(i)}\mathbf{b}^{(i)}) \\
&= (\mathbf{y} - \mathbf{y}^{(i)}) - (\mathbf{X} - \mathbf{X}^{(i)})\mathbf{b}^{(i)} - \mathbf{X}(\mathbf{b} - \mathbf{b}^{(i)}) \\
&= y_i\,\mathbf{v}^{(i)} - \mathbf{v}^{(i)}\mathbf{x}_i{}'\mathbf{b}^{(i)} - \mathbf{X}(\mathbf{b} - \mathbf{b}^{(i)}) \qquad (\text{since } \mathbf{y} - \mathbf{y}^{(i)} = y_i\,\mathbf{v}^{(i)},\ \mathbf{X} - \mathbf{X}^{(i)} = \mathbf{v}^{(i)}\mathbf{x}_i{}').
\end{aligned}$$

Thus we have shown

$$\mathbf{e} - \mathbf{e}^{(i)} = u_i\,\mathbf{v}^{(i)} - \mathbf{X}(\mathbf{b} - \mathbf{b}^{(i)}), \tag{3}$$

where

$$u_i \equiv y_i - \mathbf{x}_i{}'\mathbf{b}^{(i)}. \tag{4}$$

Multiply both sides of (3) from left by $\mathbf{M}$ ($\equiv \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$). The LHS of the equation that results is $\mathbf{Me} - \mathbf{Me}^{(i)}$. The RHS is $u_i \mathbf{Mv}^{(i)}$ since $\mathbf{MX} = \mathbf{0}$. We know from Section 1.2 that $\mathbf{Me} = \mathbf{e}$. Also,

$$\mathbf{Me}^{(i)} = \mathbf{e}^{(i)} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{e}^{(i)} = \mathbf{e}^{(i)} \qquad \text{(by (b) above).} \tag{5}$$

Thus $\mathbf{Me} - \mathbf{Me}^{(i)} = \mathbf{e} - \mathbf{e}^{(i)}$. Therefore, pre-multiplication of (3) by $\mathbf{M}$ yields:

$$\mathbf{e} - \mathbf{e}^{(i)} = u_i \, \mathbf{Mv}^{(i)}. \tag{6}$$

Multiplying both sides of this last equation by $\mathbf{v}^{(i)\prime}$, we obtain

$$\mathbf{v}^{(i)\prime}\mathbf{e} - \mathbf{v}^{(i)\prime}\mathbf{e}^{(i)} = u_i \, \mathbf{v}^{(i)\prime}\mathbf{Mv}^{(i)}. \tag{7}$$

The first term on the LHS is e_i, the i-th element of $\mathbf{e}$. The second term on the LHS is zero by (a) above. Therefore,

$$e_i = u_i \, \mathbf{v}^{(i)\prime}\mathbf{Mv}^{(i)}. \tag{8}$$

The $\mathbf{v}^{(i)\prime}\mathbf{Mv}^{(i)}$ on the RHS of this equation can be written as follows:

$$\mathbf{v}^{(i)\prime}\mathbf{Mv}^{(i)} = \mathbf{v}^{(i)\prime}\mathbf{v}^{(i)} - \mathbf{v}^{(i)\prime}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\,\mathbf{Xv}^{(i)} \qquad (\text{since } \mathbf{M} \equiv \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}') \tag{9}$$

$$= 1 - \mathbf{x}_i{}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_i, \qquad (\text{since } \mathbf{X}'\mathbf{v}^{(i)} = \mathbf{x}_i) \tag{10}$$

which by definition is $1 - p_i$ (see (1.2.20)). Therefore, we have shown:

$$u_i \; (\equiv y_i - \mathbf{x}_i{}'\mathbf{b}^{(i)}) = \frac{e_i}{1 - p_i}. \tag{11}$$

Next, multiply both sides of (3) by $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. Since $\mathbf{X}'\mathbf{e} = \mathbf{0}$ by construction and since $\mathbf{X}'\mathbf{e}^{(i)} = \mathbf{0}$ by (b) above, the LHS of the equation that results is $\mathbf{0}$. Since $\mathbf{X}'\mathbf{v}^{(i)} = \mathbf{x}_i$, the RHS of the equation is: $u_i \, (\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_i - (\mathbf{b} - \mathbf{b}^{(i)})$. Therefore,

$$\mathbf{0} = u_i \, (\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_i - (\mathbf{b} - \mathbf{b}^{(i)}). \tag{12}$$

Combining (11) and (12), we arrive at the desired conclusion (1.2.19).

Proof 2.

The second proof utilizes the following result (famous in numerical analysis):

> **Sherman-Morrison Formula:** Let $\mathbf{A}$ be an $n \times n$ invertible matrix and let $\mathbf{c}, \mathbf{d}$ be n-vectors such that $1 + \mathbf{d}'\mathbf{Ac} \neq 0$. Then $\mathbf{A} + \mathbf{cd}'$ is invertible with the inverse given by
>
> $$(\mathbf{A} + \mathbf{cd}')^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{cd}'\mathbf{A}^{-1}}{1 + \mathbf{d}'\mathbf{Ac}}.$$

(This can be verified easily by noting that $\mathbf{A}^{-1}\mathbf{c}\mathbf{d}'\mathbf{A}^{-1}\mathbf{c}\mathbf{d}' = \mathbf{d}'\mathbf{A}^{-1}\mathbf{c}\mathbf{A}^{-1}\mathbf{c}\mathbf{d}'$.)

With this lemma, the desired result can be derived directly, as follows.

$$
\begin{aligned}
\mathbf{b}^{(i)} &= \left(\sum_{j\neq i} \mathbf{x}_j \mathbf{x}_j'\right)^{-1} \sum_{j\neq i} \mathbf{x}_j \cdot y_j \qquad \text{(by definition of } \mathbf{b}^{(i)}\text{)} \\
&= \left(\sum_{j=1}^{n} \mathbf{x}_j \mathbf{x}_j' - \mathbf{x}_i \mathbf{x}_i'\right)^{-1} \left(\sum_{j=1}^{n} \mathbf{x}_j \cdot y_j - \mathbf{x}_j \cdot y_j\right) \\
&= (\mathbf{X}'\mathbf{X} - \mathbf{x}_i\mathbf{x}_i')^{-1} (\mathbf{X}'\mathbf{y} - \mathbf{x}_i \cdot y_i) \\
&= \left[(\mathbf{X}'\mathbf{X})^{-1} + \frac{(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_i\mathbf{x}_i'(\mathbf{X}'\mathbf{X})^{-1}}{1 - \mathbf{x}_i'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_i}\right] (\mathbf{X}'\mathbf{y} - \mathbf{x}_i \cdot y_i) \\
&\qquad \text{(by setting } \mathbf{A} = \mathbf{X}'\mathbf{X}, \mathbf{c} = -\mathbf{x}_i, \mathbf{d} = \mathbf{x}_i \text{ in the Sherman-Morrison Formula)} \\
&= \left[(\mathbf{X}'\mathbf{X})^{-1} + \frac{(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_i\mathbf{x}_i'(\mathbf{X}'\mathbf{X})^{-1}}{1 - p_i}\right] (\mathbf{X}'\mathbf{y} - \mathbf{x}_i \cdot y_i) \qquad \text{(since } p_i \equiv \mathbf{x}_i'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_i\text{)} \\
&= \mathbf{b} - \frac{(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_i \cdot (y_i - \mathbf{x}_i'\mathbf{b})}{1 - p_i} \qquad \text{(this you can derive through a series of multiplications)} \\
&= \mathbf{b} - \frac{1}{1 - p_i}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_i \cdot e_i \qquad \text{(since } e_i \equiv y_i - \mathbf{x}_i'\mathbf{b}\text{).}
\end{aligned}
$$