Proof that "$\mathrm{E}(X^2) = 0$" $\Rightarrow$ "$\mathrm{Prob}(X = 0) = 1$"

In Chapter 7, I used the fact that "$\mathrm{Prob}(X = 0) = 1$" $\Rightarrow$ "$\mathrm{E}(X^2) = 0$" in several places (e.g., 4th line from bottom of p. 462 and Example 7.8 on p. 466). This is obvious because probability zero events don't affect expected values. What several students wanted to know is whether the converse is true. Here I prove that the converse is indeed true. My proof uses **Markov's Inequality**:

> Suppose that Y is a random variable such that $\mathrm{Prob}(Y \geq 0) = 1$. Then for any given number $\varepsilon > 0$,
> $$\mathrm{Prob}(Y \geq \varepsilon) \leq \frac{\mathrm{E}(Y)}{\varepsilon}.$$

[Incidentally, setting $Y = [X - \mathrm{E}(X)]^2$ produces the celebrated Chebychev's inequality that $\mathrm{Prob}(|X - \mathrm{E}(X)| \geq \varepsilon) \leq \mathrm{Var}(X)/\varepsilon^2$.]

Here is my proof that "$\mathrm{E}(X^2) = 0$" $\Rightarrow$ "$\mathrm{Prob}(X = 0) = 1$" (i.e., "$\mathrm{Prob}(X \neq 0) = 0$"). Set $Y = X^2$ and $\varepsilon = 1/n^2$ in Markov's inequality to obtain

$$\mathrm{Prob}(|X| \geq \frac{1}{n}) \leq n^2\,\mathrm{E}(X^2),\ n = 1, 2, ...$$

Since $\mathrm{E}(X^2) = 0$, this implies

$$\mathrm{Prob}(|X| \geq \frac{1}{n}) = 0,\ n = 1, 2, ... \tag{$*$}$$

Hence $\mathrm{Prob}(|X| > 0) = 0$.

This ends the proof because $\mathrm{Prob}(|X| > 0) = \mathrm{Prob}(X \neq 0)$, but the last step (claiming that $\mathrm{Prob}(|X| > 0) = 0$ from $(*)$ actually requires a more subtle argument if you want to be rigorous. Let A_n be the event (a subset of the "sample space" $\mathbf{\Omega}$) such that $|X| \geq \frac{1}{n}$ (so $\mathrm{Prob}(A_n) = \mathrm{Prob}(|X| \geq \frac{1}{n})$) and let A_∞ be the event such that $|X| > 0$ (so $\mathrm{Prob}(A_\infty) = \mathrm{Prob}(|X| > 0)$). Then $A_n \subset A_{n+1} \subset \cdots$ and $\bigcup_{n=1}^{\infty} A_n = A_\infty$. So by the axioms of the probability measure, $\lim_{n\to\infty} \mathrm{Prob}(A_n) = \mathrm{Prob}(A_\infty)$. But by $(*)$ we we have $\lim_{n\to\infty} \mathrm{Prob}(A_n) = 0$.