Proof of Matrix Inequality Cited in Chapter 8 Analytical Exercise 1

In the hint to Analytical Exercise 1 of Chapter 8, there is a matrix inequality on p. 552 that $|\mathbf{A}+\mathbf{B}| \geq |\mathbf{A}|$ if $\mathbf{A}$ and $\mathbf{B}$ are (symmetric and) positive semi-definite. Here is a proof. The proof uses the following result (see, e.g., Amemiya (1985, Theorem 11, p. 460)):

> Let $\mathbf{A}$ and $\mathbf{B}$ be symmetric matrices $(n \times n)$ (so their eigenvalues are all real). Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $\mathbf{A}$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the eigenvalues of $\mathbf{A}+\mathbf{B}$. Then $\mu_i \geq \lambda_i$, $i = 1, 2, \ldots, n$, if $\mathbf{B}$ is positive semi-definite (i.e., nonnegative definite).

Therefore, if both $\mathbf{A}$ and $\mathbf{B}$ are positive semi-definite, then $\mu_i \geq \lambda_i \geq 0$ for $i = 1, 2, \ldots, n$. Since the determinant of a matrix is the product of its eigenvalues (see, e.g., Amemiya (1985, Theorem 2, p. 459)), we have

$$|\mathbf{A}+\mathbf{B}| = \prod_{i=1}^{n} \mu_i \geq \prod_{i=1}^{n} \lambda_i = |\mathbf{A}|.$$