Solution to Chapter 9 Analytical Exercises

1. From the hint, we have

$$\frac{1}{T}\sum_{t=1}^{T}\Delta\xi_t\cdot\xi_{t-1}=\frac{1}{2}\left(\frac{\xi_T}{\sqrt{T}}\right)^2-\frac{1}{2}\left(\frac{\xi_0}{\sqrt{T}}\right)^2-\frac{1}{2T}\sum_{t=1}^{T}(\Delta\xi_t)^2. \qquad (*)$$

Consider the second term on the RHS of $(*)$. Since $\text{E}(\xi_0/\sqrt{T})\to 0$ and $\text{Var}(\xi_0/\sqrt{T})\to 0$, $\xi_0/\sqrt{T}$ converges in mean square (by Chevychev's LLN), and hence in probability, to 0. So the second term vanishes (converges in probability to zero) (this can actually be shown directly from the definition of convergence in probability). Next, consider the expression $\xi_T/\sqrt{T}$ in the first term on the RHS of $(*)$. It can be written as

$$\frac{\xi_T}{\sqrt{T}}=\frac{1}{\sqrt{T}}(\xi_0+\Delta\xi_1+\cdots+\Delta\xi_T)=\frac{\xi_0}{\sqrt{T}}+\sqrt{T}\frac{1}{T}\sum_{t=1}^{T}\Delta\xi_t.$$

As just seen, $\frac{\xi_0}{\sqrt{T}}$ vanishes. Since $\Delta\xi_t$ is I(0) satisfying (9.2.1)-(9.2.3), the hypothesis of Proposition 6.9 is satisfied (in particular, the absolute summability in the hypothesis of the Proposition is satisfied because it is implied by the one-summability (9.2.3a)). So

$$\sqrt{T}\frac{1}{T}\sum_{t=1}^{T}\Delta\xi_t\underset{\text{d}}{\to}\lambda X,\ \ X\sim N(0,1).$$

where λ^2 is the long-run variance of $\Delta\xi_t$. Regarding the third term on the RHS of $(*)$, since $\Delta\xi_t$ is ergodic stationary, $\frac{1}{2T}\sum_{t=1}^{T}(\Delta\xi_t)^2$ converges in probability to $\frac{1}{2}\gamma_0$. Finally, by Lemma 2.4(a) we conclude that the RHS of $(*)$ converges in distribution to $\frac{\lambda^2}{2}X^2-\frac{1}{2}\gamma_0$.

2. (a) The hint is the answer.

   (b) From (a),

$$T\cdot(\widehat{\rho}^{\mu}-1)=\frac{\frac{1}{T}\sum_{t=1}^{T}\Delta y_t\,y_{t-1}^{\mu}}{\frac{1}{T^2}\sum_{t=1}^{T}(y_{t-1}^{\mu})^2}.$$

   Apply Proposition 9.2(d) to the numerator and Proposition 9.2(c) to the denominator.

   (c) Since $\{y_t\}$ is random walk, $\lambda^2=\gamma_0$. Just set $\lambda^2=\gamma_0$ in (4) of the question.

   (d) • First, a proof that $\widehat{\alpha}^*\to_{\text{p}}0$. By the algebra of OLS,

$$\begin{aligned}\widehat{\alpha}^*&=\frac{1}{T}\sum_{t=1}^{T}(y_t-\widehat{\rho}^{\mu}y_{t-1})\\&=\frac{1}{T}\sum_{t=1}^{T}(\Delta y_t-(\widehat{\rho}^{\mu}-1)y_{t-1})\\&=\frac{1}{T}\sum_{t=1}^{T}\Delta y_t-(\widehat{\rho}^{\mu}-1)\frac{1}{T}\sum_{t=1}^{T}y_{t-1}\\&=\frac{1}{T}\sum_{t=1}^{T}\Delta y_t-\frac{1}{\sqrt{T}}\Big(T\cdot(\widehat{\rho}^{\mu}-1)\Big)\left(\frac{1}{\sqrt{T}}\frac{1}{T}\sum_{t=1}^{T}y_{t-1}\right).\end{aligned}$$

The first term after the last equality, $\frac{1}{T}\sum_{t=1}^{T}\Delta y_t$, vanishes (converges to zero in probability) because Δy_t is ergodic stationary and $\mathrm{E}(\Delta y_t) = 0$. To show that the second term after the last equality vanishes, we first note that $\frac{1}{\sqrt{T}}\Big(T\cdot(\widehat{\rho}^{\mu}-1)\Big)$ vanishes because $T\cdot(\widehat{\rho}^{\mu}-1)$ converges to a random variable by (b). By (6) in the hint, $\frac{1}{\sqrt{T}}\frac{1}{T}\sum_{t=1}^{T} y_{t-1}$ converges to a random variable. Therefore, by Lemma 2.4(b), the whole second term vanishes.

- Now turn to s^2. From the hint,

$$s^2 = \frac{1}{T-1}\sum_{t=1}^{T}(\Delta y_t - \widehat{\alpha}^*)^2 - \frac{2}{T-1}\cdot[T\cdot(\widehat{\rho}^{\mu}-1)]\cdot\frac{1}{T}\sum_{t=1}^{T}(\Delta y_t - \widehat{\alpha}^*)\cdot y_{t-1}$$
$$+\frac{1}{T-1}\cdot[T\cdot(\widehat{\rho}^{\mu}-1)]^2\cdot\frac{1}{T^2}\sum_{t=1}^{T}(y_{t-1})^2. \qquad (*)$$

Since $\widehat{\alpha}^* \to_{\mathrm{p}} 0$, it should be easy to show that the first term on the RHS of $(*)$ converges to γ_0 in probability. Regarding the second term, rewrite it as

$$\frac{2}{T-1}\cdot[T\cdot(\widehat{\rho}^{\mu}-1)]\cdot\frac{1}{T}\sum_{t=1}^{T}\Delta y_t\, y_{t-1} - \frac{2\sqrt{T}}{T-1}\cdot[T\cdot(\widehat{\rho}^{\mu}-1)]\cdot\widehat{\alpha}^*\cdot\frac{1}{\sqrt{T}}\frac{1}{T}\sum_{t=1}^{T} y_{t-1}. \quad (**)$$

By Proposition 9.2(b), $\frac{1}{T}\sum_{t=1}^{T}\Delta y_t\, y_{t-1}$ converges to a random variable. So does $T\cdot(\widehat{\rho}^{\mu}-1)$. Hence the first term of $(**)$ vanishes. Turning to the second term of $(**)$, (6) in the question means $\frac{1}{\sqrt{T}}\frac{1}{T}\sum_{t=1}^{T} y_{t-1}$ converges to a random variable. It should now be routine to show that the whole second term of $(**)$ vanishes. A similar argument, this time utilizing Proposition 9.2(a), shows that the third term of $(*)$ vanishes.

(e) By (7) in the hint and (3), a little algebra yields

$$t^{\mu} = \frac{\widehat{\rho}^{\mu}-1}{s\cdot\frac{1}{\sqrt{\sum_{t=1}^{T}(y_{t-1}^{\mu})^2}}} = \frac{\frac{1}{T}\sum_{t=1}^{T}\Delta y_t\, y_{t-1}^{\mu}}{s\cdot\sqrt{\frac{1}{T^2}\sum_{t=1}^{T}(y_{t-1}^{\mu})^2}}.$$

Use Proposition 9.2(c) and (d) with $\lambda^2 = \gamma_0 = \sigma^2$ and the fact that s is consistent for σ to complete the proof.

3. (a) The hint is the answer.

(b) From (a), we have

$$T\cdot(\widehat{\rho}^{\tau}-1) = \frac{\frac{1}{T}\sum_{t=1}^{T}\Delta y_t\, y_{t-1}^{\tau}}{\frac{1}{T^2}\sum_{t=1}^{T}(y_{t-1}^{\tau})^2}.$$

Let ξ_t and ξ_t^{τ} be as defined in the hint. Then $\Delta y_t = \delta + \Delta\xi_t$ and $y_t^{\tau} = \xi_t^{\tau}$. By construction, $\sum_{t=1}^{T} y_{t-1}^{\tau} = 0$. So

$$T\cdot(\widehat{\rho}^{\tau}-1) = \frac{\frac{1}{T}\sum_{t=1}^{T}\Delta\xi_t\, \xi_{t-1}^{\tau}}{\frac{1}{T^2}\sum_{t=1}^{T}(\xi_{t-1}^{\tau})^2}.$$

Since $\{\xi_t\}$ is driftless I(1), Proposition 9.2(e) and (f) can be used here.

(c) Just observe that $\lambda^2 = \gamma_0$ if $\{y_t\}$ is a random walk with or without drift.

4. From the hint,

$$\frac{1}{T}\sum_{t=1}^{T} y_{t-1}\,\varepsilon_t = \psi(1)\frac{1}{T}\sum_{t=1}^{T} w_{t-1}\,\varepsilon_t + \frac{1}{T}\sum_{t=1}^{T}\eta_{t-1}\,\varepsilon_t + (y_0-\eta_0)\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t. \qquad (*)$$

Consider first the second term on the RHS of $(*)$. Since η_{t-1}, which is a function of $(\varepsilon_{t-1}, \varepsilon_{t-2}, \dots)$, is independent of ε_t, we have: $\mathrm{E}(\eta_{t-1}\varepsilon_t) = \mathrm{E}(\eta_{t-1})\,\mathrm{E}(\varepsilon_t) = 0$. Then by the ergodic theorem this second term vanishes. Regarding the third term of $(*)$, $\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t \to_{\mathrm{p}} 0$. So the whole third term vanishes. Lastly, consider the first term on the RHS of $(*)$. Since $\{w_t\}$ is random walk and $\varepsilon_t = \Delta w_t$, Proposition 9.2(b) with $\lambda^2 = \gamma_0 = \sigma^2$ implies $\frac{1}{T}\sum_{t=1}^{T} w_{t-1}\,\varepsilon_t \to_{\mathrm{d}} \left(\frac{\sigma^2}{2}\right)[W(1)^2 - 1]$.

5. Comparing Proposition 9.6 and 9.7, the null is the same (that $\{\Delta y_t\}$ is zero-mean stationary AR(p), $\phi(L)\Delta y_t = \varepsilon_t$, whose MA representation is $\Delta y_t = \psi(L)\varepsilon_t$ with $\psi(L) \equiv \phi(L)^{-1}$) but the augmented autoregression in Proposition 9.7 has an intercept. The proof of Proposition 9.7 (for $p = 1$) makes appropriate changes on the argument developed on pp. 587-590. Let $\mathbf{b}$ and $\boldsymbol{\beta}$ be as defined in the hint. The $\mathbf{A}_T$ and $\mathbf{c}_T$ for the present case is

$$\mathbf{A}_T = \begin{bmatrix} \frac{1}{T^2}\sum_{t=1}^{T}(y_{t-1}^{\mu})^2 & \frac{1}{\sqrt{T}}\,\frac{1}{T}\sum_{t=1}^{T} y_{t-1}^{\mu}\,(\Delta y_{t-1})^{(\mu)} \\ \frac{1}{\sqrt{T}}\,\frac{1}{T}\sum_{t=1}^{T}(\Delta y_{t-1})^{(\mu)}\,y_{t-1}^{\mu} & \frac{1}{T}\sum_{t=1}^{T}[(\Delta y_{t-1})^{(\mu)}]^2 \end{bmatrix},$$

$$\mathbf{c}_T = \begin{bmatrix} \frac{1}{T}\sum_{t=1}^{T} y_{t-1}^{\mu}\,\varepsilon_t^{\mu} \\ \frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\Delta y_{t-1})^{(\mu)}\,\varepsilon_t^{\mu} \end{bmatrix} = \begin{bmatrix} \frac{1}{T}\sum_{t=1}^{T} y_{t-1}^{\mu}\,\varepsilon_t \\ \frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\Delta y_{t-1})^{(\mu)}\,\varepsilon_t \end{bmatrix},$$

where ε_t^{μ} is the residual from the regression of ε_t on a constant for $t = 1, 2, ..., T$.

- (1,1) element of $\mathbf{A}_T$: Since $\{y_t\}$ is driftless I(1) under the null, Proposition 9.2(c) can be used to claim that $\frac{1}{T^2}\sum_{t=1}^{T}(y_{t-1}^{\mu})^2 \to_{\mathrm{d}} \lambda^2\int(W^{\mu})^2$, where $\lambda^2 = \sigma^2[\psi(1)]^2$ with $\sigma^2 \equiv \mathrm{Var}(\varepsilon_t)$.
- (2,2) element of $\mathbf{A}_T$: Since $(\Delta y_{t-1})^{(\mu)} = \Delta y_{t-1} - \frac{1}{T}\sum_{t=1}^{T}\Delta y_{t-1}$, this element can be written as

$$\frac{1}{T}\sum_{t=1}^{T}[(\Delta y_{t-1})^{(\mu)}]^2 = \frac{1}{T}\sum_{t=1}^{T}(\Delta y_{t-1})^2 - \left(\frac{1}{T}\sum_{t=1}^{T}\Delta y_{t-1}\right)^2.$$

  Since $\mathrm{E}(\Delta y_{t-1}) = 0$ and $\mathrm{E}[(\Delta y_{t-1})^2] = \gamma_0$ (the variance of Δy_t), this expression converges in probability to γ_0.
- Off diagonal elements of $\mathbf{A}_T$: it equals

$$\frac{1}{\sqrt{T}}\frac{1}{T}\sum_{t=1}^{T}(\Delta y_{t-1})^{(\mu)}\,y_{t-1}^{\mu} = \frac{1}{\sqrt{T}}\left[\frac{1}{T}\sum_{t=1}^{T}(\Delta y_{t-1})\,y_{t-1}\right] - \left(\frac{1}{\sqrt{T}}\frac{1}{T}\sum_{t=1}^{T} y_{t-1}\right)\left(\frac{1}{T}\sum_{t=1}^{T}\Delta y_{t-1}\right).$$

  The term in the square bracket is (9.4.14), which is shown to converge to a random variable (Review Question 3 of Section 9.4). The next term, $\frac{1}{\sqrt{T}}\frac{1}{T}\sum_{t=1}^{T} y_{t-1}$, converges to a random variable by (6) assumed in Analytical Exercise 2(d). The last term, $\frac{1}{T}\sum_{t=1}^{T}\Delta y_{t-1}$, converges to zero in probability. Therefore, the off-diagonal elements vanish.

Taken together, we have shown that $\mathbf{A}_T$ is asymptotically diagonal:

$$\mathbf{A}_T \underset{\mathrm{d}}{\to} \begin{bmatrix} \lambda^2\cdot\int_0^1[W^{\mu}(r)]^2\,\mathrm{d}r & 0 \\ 0 & \gamma_0 \end{bmatrix},$$

so

$$(\mathbf{A}_T)^{-1} \underset{\mathrm{d}}{\to} \begin{bmatrix} \left(\lambda^2 \cdot \int_0^1 [W^\mu(r)]^2 \,\mathrm{d}r\right)^{-1} & 0 \\ 0 & \gamma_0^{-1} \end{bmatrix}.$$

Now turn to $\mathbf{c}_T$.

- 1st element of $\mathbf{c}_T$: Recall that $y_{t-1}^\mu \equiv y_{t-1} - \frac{1}{T}\sum_{t=1}^T y_{t-1}$. Combine this with the BN decomposition $y_{t-1} = \psi(1)w_{t-1} + \eta_{t-1} + (y_0 - \eta_0)$ with $w_{t-1} \equiv \varepsilon_1 + \cdots + \varepsilon_{t-1}$ to obtain

$$\frac{1}{T}\sum_{t=1}^T y_{t-1}^\mu \,\varepsilon_t = \psi(1)\frac{1}{T}\sum_{t=1}^T w_{t-1}^\mu \,\varepsilon_t + \frac{1}{T}\sum_{t=1}^T \eta_{t-1}^\mu \,\varepsilon_t,$$

where $w_{t-1}^\mu \equiv w_{t-1} - \frac{1}{T}\sum_{t=1}^T w_{t-1}$. η_{t-1}^μ is defined similarly. Since η_{t-1} is independent of ε_t, the second term on the RHS vanishes. Noting that $\Delta w_t = \varepsilon_t$ and applying Proposition 9.2(d) to the random walk $\{w_t\}$, we obtain

$$\frac{1}{T}\sum_{t=1}^T w_{t-1}^\mu \,\varepsilon_t \underset{\mathrm{d}}{\to} \left(\frac{\sigma^2}{2}\right)\left\{[W(1)^\mu]^2 - [W(0)^\mu]^2 - 1\right\}.$$

Therefore, the 1st element of $\mathbf{c}_T$ converges in distribution to

$$c_1 \equiv \sigma^2 \cdot \psi(1) \cdot \frac{1}{2}\left\{[W(1)^\mu]^2 - [W(0)^\mu]^2 - 1\right\}.$$

- 2nd element of $\mathbf{c}_T$: Using the definition $(\Delta y_{t-1})^{(\mu)} \equiv \Delta y_{t-1} - \frac{1}{T}\sum_{t=1}^T \Delta y_{t-1}$, it should be easy to show that it converges in distribution to

$$c_2 \sim N(0, \gamma_0 \cdot \sigma^2).$$

Using the results derived so far, the modification to be made on (9.4.20) and (9.4.21) on p. 590 for the present case where the augmented autoregression has an intercept is

$$T \cdot (\widehat{\rho}^\mu - 1) \underset{\mathrm{d}}{\to} \frac{\sigma^2\psi(1)}{\lambda^2} \cdot \frac{\frac{1}{2}\left\{[W(1)^\mu]^2 - [W(0)^\mu]^2 - 1\right\}}{\int_0^1 [W^\mu(r)]^2 \,\mathrm{d}r} \quad \text{or} \quad \frac{\lambda^2}{\sigma^2\psi(1)} \cdot T \cdot (\widehat{\rho}^\mu - 1) \underset{\mathrm{d}}{\to} DF_\rho^\mu,$$

$$\sqrt{T} \cdot (\widehat{\zeta}_1 - \zeta_1) \underset{\mathrm{d}}{\to} N\Big(0, \frac{\sigma^2}{\gamma_0}\Big).$$

Repeating exactly the same argument that is given in the subsection entitled "Deriving Test Statistics" on p. 590, we can claim that $\frac{\lambda^2}{\sigma^2\psi(1)}$ is consistently estimated by $1/(1 - \widehat{\zeta})$. This completes the proof of claim (9.4.34) of Proposition 9.7.

6. (a) The hint is the answer.
   (b) The proof should be straightforward.

7. The one-line proof displayed in the hint is (with i replaced by k to avoid confusion)

$$\sum_{j=0}^\infty |\alpha_j| = \sum_{j=0}^\infty \left| -\sum_{k=j+1}^\infty \psi_k \right| \le \sum_{j=0}^\infty \sum_{k=j+1}^\infty |\psi_k| = \sum_{k=0}^\infty k|\psi_k| < \infty, \qquad (*)$$

where $\{\psi_k\}$ $(k = 0, 1, 2, ...)$ is one-summable as assumed in (9.2.3a). We now justify each of the equalities and inequalities. For this purpose, we reproduce here the facts from calculus shown on pp. 429-430:

(i) If $\{a_k\}$ is absolutely summable, then $\{a_k\}$ is summable (i.e., $-\infty < \sum_{k=0}^{\infty} a_k < \infty$) and

$$\left|\sum_{k=0}^{\infty} a_k\right| \leq \sum_{k=0}^{\infty} |a_k|.$$

(ii) Consider a sequence with two subscripts, $\{a_{jk}\}$ $(j, k = 0, 1, 2, \ldots)$. Suppose $\sum_{j=0}^{\infty} |a_{jk}| < \infty$ for each k and let $s_k \equiv \sum_{j=0}^{\infty} |a_{jk}|$. Suppose $\{s_k\}$ is summable. Then

$$\left|\sum_{j=0}^{\infty}\Big(\sum_{k=0}^{\infty} a_{jk}\Big)\right| < \infty \quad \text{and} \quad \sum_{j=0}^{\infty}\Big(\sum_{k=0}^{\infty} a_{jk}\Big) = \sum_{k=0}^{\infty}\Big(\sum_{j=0}^{\infty} a_{jk}\Big) < \infty.$$

Since $\{\psi_k\}$ is one-summable, it is absolutely summable. Let

$$a_k = \begin{cases} \psi_k & \text{if } k \geq j+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{a_k\}$ is absolutely summable because $\{\psi_k\}$ is absolutely summable. So by (i) above, we have

$$\left|-\sum_{k=j+1}^{\infty} \psi_k\right| = \left|\sum_{k=j+1}^{\infty} \psi_k\right| = \left|\sum_{k=0}^{\infty} a_k\right| \leq \sum_{k=0}^{\infty} |a_k| = \sum_{k=j+1}^{\infty} |\psi_k|.$$

Summing over $j = 0, 1, 2, ..., n$, we obtain

$$\sum_{j=0}^{n}\left|-\sum_{k=j+1}^{\infty} \psi_k\right| \leq \sum_{j=0}^{n}\sum_{k=j+1}^{\infty} |\psi_k|.$$

If the limit as $n \to \infty$ of the RHS exists and is finite, then the limit of the LHS exists and is finite (this follows from the fact that if $\{x_n\}$ is non-decreasing in n and if $x_n \leq A < \infty$, then the limit of x_n exists and is finite; set $x_n \equiv \sum_{j=0}^{n} | - \sum_{k=j+1}^{\infty} \psi_k|$). Thus, provided that $\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty} |\psi_k|$ is well-defined, we have

$$\sum_{j=0}^{\infty}\left|-\sum_{k=j+1}^{\infty} \psi_k\right| \leq \sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty} |\psi_k|.$$

We now show that $\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty} |\psi_k|$ is well-defined. In (ii), set a_{jk} as

$$a_{jk} = \begin{cases} |\psi_k| & \text{if } k \geq j+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sum_{j=0}^{\infty} |a_{jk}| = k\,|\psi_k| < \infty$ for each k and $s_k = k\,|\psi_k|$. By one-summability of $\{\psi_k\}$, $\{s_k\}$ is summable. So the conditions in (ii) are satisfied for this choice of a_{jk}. We therefore conclude that

$$\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty} |\psi_k| = \sum_{j=0}^{\infty}\Big(\sum_{k=0}^{\infty} a_{jk}\Big) = \sum_{k=0}^{\infty}\Big(\sum_{j=0}^{\infty} a_{jk}\Big) = \sum_{k=0}^{\infty} k\,|\psi_k| < \infty.$$

This completes the proof.