Solution to Chapter 8 Analytical Exercises

1. From the hint,

$$\sum_{t=1}^{n}(\mathbf{y}_t - \mathbf{\Pi}'\mathbf{x}_t)(\mathbf{y}_t - \mathbf{\Pi}'\mathbf{x}_t)' = \sum_{t=1}^{n}\widehat{\mathbf{v}}_t\widehat{\mathbf{v}}_t' + (\widehat{\mathbf{\Pi}} - \mathbf{\Pi})'\Big(\sum_{t=1}^{n}\mathbf{x}_t\mathbf{x}_t'\Big)(\widehat{\mathbf{\Pi}} - \mathbf{\Pi}).$$

   But

$$(\widehat{\mathbf{\Pi}} - \mathbf{\Pi})'\Big(\sum_{t=1}^{n}\mathbf{x}_t\mathbf{x}_t'\Big)(\widehat{\mathbf{\Pi}} - \mathbf{\Pi}) = \sum_{t=1}^{n}\Big[\widehat{\mathbf{\Pi}} - \mathbf{\Pi})'\mathbf{x}_t\mathbf{x}_t'(\widehat{\mathbf{\Pi}} - \mathbf{\Pi})\Big]$$

   is positive semi-definite.

2. Since $\mathbf{y}_t = \mathbf{\Pi}_0'\mathbf{x}_t + \mathbf{v}_t$, we have $\mathbf{y}_t - \mathbf{\Pi}'\mathbf{x}_t = \mathbf{v}_t + (\mathbf{\Pi}_0 - \mathbf{\Pi})'\mathbf{x}_t$. So

$$\begin{aligned}
&\mathrm{E}[(\mathbf{y}_t - \mathbf{\Pi}'\mathbf{x}_t)(\mathbf{y}_t - \mathbf{\Pi}'\mathbf{x}_t)'] = \mathrm{E}[(\mathbf{v}_t + (\mathbf{\Pi}_0 - \mathbf{\Pi})'\mathbf{x}_t)(\mathbf{v}_t + (\mathbf{\Pi}_0 - \mathbf{\Pi})'\mathbf{x}_t)'] \\
&= \mathrm{E}(\mathbf{v}_t\mathbf{v}_t') + \mathrm{E}[\mathbf{v}_t\mathbf{x}_t'(\mathbf{\Pi}_0 - \mathbf{\Pi})] + \mathrm{E}[(\mathbf{\Pi}_0 - \mathbf{\Pi})'\mathbf{x}_t\mathbf{v}_t'] + (\mathbf{\Pi}_0 - \mathbf{\Pi})'\,\mathrm{E}(\mathbf{x}_t\mathbf{x}_t')(\mathbf{\Pi}_0 - \mathbf{\Pi}) \\
&= \mathrm{E}(\mathbf{v}_t\mathbf{v}_t') + (\mathbf{\Pi}_0 - \mathbf{\Pi})'\,\mathrm{E}(\mathbf{x}_t\mathbf{x}_t')(\mathbf{\Pi}_0 - \mathbf{\Pi}) \quad (\text{since } \mathrm{E}(\mathbf{x}_t\mathbf{v}_t') = \mathbf{0}).
\end{aligned}$$

   So

$$\widehat{\mathbf{\Omega}}(\mathbf{\Pi}) \to \mathbf{\Omega}_0 + (\mathbf{\Pi}_0 - \mathbf{\Pi})'\,\mathrm{E}(\mathbf{x}_t\mathbf{x}_t')(\mathbf{\Pi}_0 - \mathbf{\Pi})$$

   almost surely. By the matrix algebra result cited in the previous question,

$$|\mathbf{\Omega}_0 + (\mathbf{\Pi}_0 - \mathbf{\Pi})'\,\mathrm{E}(\mathbf{x}_t\mathbf{x}_t')(\mathbf{\Pi}_0 - \mathbf{\Pi})| \geq |\mathbf{\Omega}_0| > 0.$$

   So for sufficiently large n, $\widehat{\mathbf{\Omega}}(\mathbf{\Pi})$ is positive definite.

3. (a) Multiply both sides of $\mathbf{z}_{tm}' = \big(\mathbf{y}_t'\mathbf{S}_m' \,\vdots\, \mathbf{x}_t'\mathbf{C}_m'\big)$ from left by $\mathbf{x}_t$ to obtain

$$\mathbf{x}_t\mathbf{z}_{tm}' = \big[\mathbf{x}_t\mathbf{y}_t'\mathbf{S}_m' \,\vdots\, \mathbf{x}_t\mathbf{x}_t'\mathbf{C}_m'\big]. \qquad (*)$$

   Do the same to the reduced form $\mathbf{y}_t' = \mathbf{x}_t'\mathbf{\Pi}_0 + \mathbf{v}_t'$ to obtain $\mathbf{x}_t\mathbf{y}_t' = \mathbf{x}_t\mathbf{x}_t'\mathbf{\Pi}_0 + \mathbf{x}_t\mathbf{v}_t'$. Substitute this into $(*)$ to obtain

$$\mathbf{x}_t\mathbf{z}_{tm}' = \big[\mathbf{x}_t\mathbf{x}_t'\mathbf{\Pi}_0\mathbf{S}_m' \,\vdots\, \mathbf{x}_t\mathbf{x}_t'\mathbf{C}_m'\big] + \big[\mathbf{x}_t\mathbf{v}_t' \,\vdots\, \mathbf{0}\big] = \mathbf{x}_t\mathbf{x}_t'\big[\mathbf{\Pi}_0\mathbf{S}_m' \,\vdots\, \mathbf{C}_m'\big] + \big[\mathbf{x}_t\mathbf{v}_t' \,\vdots\, \mathbf{0}\big].$$

   Take the expected value of both sides and use the fact that $\mathrm{E}(\mathbf{x}_t\mathbf{v}_t') = \mathbf{0}$ to obtain the desired result.

   (b) Use the reduced form $\mathbf{y}_t = \mathbf{\Pi}_0'\mathbf{x}_t + \mathbf{v}_t$ to derive

$$\mathbf{y}_t + \mathbf{\Gamma}^{-1}\mathbf{B}\mathbf{x}_t = \mathbf{v}_t + (\mathbf{\Pi}_0' + \mathbf{\Gamma}^{-1}\mathbf{B})\mathbf{x}_t$$

   as in the hint. So

$$\begin{aligned}
&(\mathbf{y}_t + \mathbf{\Gamma}^{-1}\mathbf{B}\mathbf{x}_t)(\mathbf{y}_t + \mathbf{\Gamma}^{-1}\mathbf{B}\mathbf{x}_t)' \\
&= [\mathbf{v}_t + (\mathbf{\Pi}_0' + \mathbf{\Gamma}^{-1}\mathbf{B})\mathbf{x}_t][\mathbf{v}_t + (\mathbf{\Pi}_0' + \mathbf{\Gamma}^{-1}\mathbf{B})\mathbf{x}_t]' \\
&= \mathbf{v}_t\mathbf{v}_t' + (\mathbf{\Pi}_0' + \mathbf{\Gamma}^{-1}\mathbf{B})\mathbf{x}_t\mathbf{v}_t' + \mathbf{v}_t\mathbf{x}_t'(\mathbf{\Pi}_0' + \mathbf{\Gamma}^{-1}\mathbf{B})' + (\mathbf{\Pi}_0' + \mathbf{\Gamma}^{-1}\mathbf{B})\mathbf{x}_t\mathbf{x}_t'(\mathbf{\Pi}_0' + \mathbf{\Gamma}^{-1}\mathbf{B})'.
\end{aligned}$$

Taking the expected value and noting that $\mathrm{E}(\mathbf{x}_t\mathbf{v}_t') = \mathbf{0}$, we obtain

$$\mathrm{E}[(\mathbf{y}_t + \boldsymbol{\Gamma}^{-1}\mathbf{B}\mathbf{x}_t)(\mathbf{y}_t + \boldsymbol{\Gamma}^{-1}\mathbf{B}\mathbf{x}_t)'] = \mathrm{E}(\mathbf{v}_t\mathbf{v}_t') + (\boldsymbol{\Pi}_0' + \boldsymbol{\Gamma}^{-1}\mathbf{B})\,\mathrm{E}(\mathbf{x}_t\mathbf{x}_t')(\boldsymbol{\Pi}_0' + \boldsymbol{\Gamma}^{-1}\mathbf{B})'.$$

Since $\{\mathbf{y}_t, \mathbf{x}_t\}$ is i.i.d., the probability limit of $\widehat{\boldsymbol{\Omega}}(\boldsymbol{\delta})$ is given by this expectation. In this expression, $\mathrm{E}(\mathbf{v}_t\mathbf{v}_t')$ equals $\boldsymbol{\Gamma}_0^{-1}\boldsymbol{\Sigma}_0(\boldsymbol{\Gamma}_0^{-1})'$ because by definition $\mathbf{v}_t \equiv \boldsymbol{\Gamma}_0^{-1}\boldsymbol{\varepsilon}_t$ and $\boldsymbol{\Sigma}_0 \equiv \mathrm{E}(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t')$.

(c) What needs to be proved is that plim $|\widehat{\boldsymbol{\Omega}}(\boldsymbol{\delta})|$ is minimized only if $\boldsymbol{\Gamma}\boldsymbol{\Pi}_0' + \mathbf{B} = \mathbf{0}$. Let $\mathbf{A} \equiv \boldsymbol{\Gamma}_0^{-1}\boldsymbol{\Sigma}_0(\boldsymbol{\Gamma}_0^{-1})'$ be the first term on the RHS of (7) and let $\mathbf{D} \equiv (\boldsymbol{\Pi}_0' + \boldsymbol{\Gamma}^{-1}\mathbf{B})\,\mathrm{E}(\mathbf{x}_t\mathbf{x}_t')(\boldsymbol{\Pi}_0' + \boldsymbol{\Gamma}^{-1}\mathbf{B})'$ be the second term. Since $\boldsymbol{\Sigma}_0$ is positive definite and $\boldsymbol{\Gamma}_0^{-1}$ is non-singular, $\mathbf{A}$ is positive definite. Since $\mathrm{E}(\mathbf{x}_t\mathbf{x}_t')$ is positive definite, $\mathbf{D}$ is positive semi-definite. Then use the following the matrix inequality (which is slightly different from the one mentioned in Analytical Exercise 1 on p. 552):

> (Theorem 22 on p. 21 of *Matrix Differential Calculus with Applications in Statistics and Econometrics* by Jan R. Magnus and Heinz Neudecker, Wiley, 1988) Let $\mathbf{A}$ be positive definite and $\mathbf{B}$ positive semi-definite. Then
>
> $$|\mathbf{A} + \mathbf{B}| \geq |\mathbf{A}|$$
>
> with equality if and only if $\mathbf{B} = \mathbf{0}$.

Hence

$$\text{plim}\,|\widehat{\boldsymbol{\Omega}}(\boldsymbol{\delta})| = |\mathbf{A} + \mathbf{D}| \geq |\mathbf{A}| = |\boldsymbol{\Gamma}_0^{-1}\boldsymbol{\Sigma}_0(\boldsymbol{\Gamma}_0^{-1})'|.$$

with equality "$|\mathbf{A} + \mathbf{D}| = |\mathbf{A}|$" only if $\mathbf{D} = \mathbf{0}$. Since $\mathrm{E}(\mathbf{x}_t\mathbf{x}_t')$ is positive definite, $\mathbf{D} \equiv (\boldsymbol{\Pi}_0' + \boldsymbol{\Gamma}^{-1}\mathbf{B})\,\mathrm{E}(\mathbf{x}_t\mathbf{x}_t')(\boldsymbol{\Pi}_0' + \boldsymbol{\Gamma}^{-1}\mathbf{B})'$ is a zero matrix only if $\boldsymbol{\Pi}_0' + \boldsymbol{\Gamma}^{-1}\mathbf{B} = \mathbf{0}$, which holds if and only if $\boldsymbol{\Gamma}\boldsymbol{\Pi}_0' + \mathbf{B} = \mathbf{0}$ since $\boldsymbol{\Gamma}$ is non-singular (the parameter space is such that $\boldsymbol{\Gamma}$ is non-singular).

(d) For $m = 1$, the LHS of (8) is

$$\boldsymbol{\alpha}_m' = \begin{bmatrix} -\gamma_{11} & 1 & -\beta_{11} & -\beta_{12} & 0 \end{bmatrix}.$$

The RHS is

$$\begin{aligned}
&\mathbf{e}_m' - \underset{(1\times(M_m+K_m))}{\boldsymbol{\delta}_m'} \begin{bmatrix} \underset{(M_m\times M)}{\mathbf{S}_m} & \underset{(M_m\times K)}{\mathbf{0}} \\ \underset{(K_m\times M)}{\mathbf{0}} & \underset{(K_m\times K)}{\mathbf{C}_m} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \gamma_{11} & \beta_{11} & \beta_{12} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.
\end{aligned}$$

(e) Since $\boldsymbol{\alpha}_m'$ is the m-th row of $[\boldsymbol{\Gamma} \vdots \mathbf{B}]$, the m-th row of of the LHS of (9) equals

$$\boldsymbol{\alpha}_m' \begin{bmatrix} \boldsymbol{\Pi}_0' \\ \mathbf{I}_K \end{bmatrix} = \left\{ \mathbf{e}_m' - \underset{(1\times(M_m+K_m))}{\boldsymbol{\delta}_m'} \begin{bmatrix} \underset{(M_m\times M)}{\mathbf{S}_m} & \underset{(M_m\times K)}{\mathbf{0}} \\ \underset{(K_m\times M)}{\mathbf{0}} & \underset{(K_m\times K)}{\mathbf{C}_m} \end{bmatrix} \right\} \begin{bmatrix} \underset{(M\times K)}{\boldsymbol{\Pi}_0'} \\ \mathbf{I}_K \end{bmatrix} \quad \text{(by (8))}$$

$$= \mathbf{e}_m' \begin{bmatrix} \boldsymbol{\Pi}_0' \\ \mathbf{I}_K \end{bmatrix} - \boldsymbol{\delta}_m' \begin{bmatrix} \mathbf{S}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{\Pi}_0' \\ \mathbf{I}_K \end{bmatrix}$$

$$= [[\boldsymbol{\Pi}_0 \vdots \mathbf{I}_K]\,\mathbf{e}_m]' - \boldsymbol{\delta}_m' \begin{bmatrix} \mathbf{S}_m\boldsymbol{\Pi}_0' \\ \mathbf{C}_m \end{bmatrix}$$

$$= \boldsymbol{\pi}_{0m}' - \boldsymbol{\delta}_m' \begin{bmatrix} \mathbf{S}_m\boldsymbol{\Pi}_0' \\ \mathbf{C}_m \end{bmatrix} \quad \text{(by the definition of } \boldsymbol{\pi}_{0m}).$$

(f) By definition (see (8.5.10)), $\boldsymbol{\Gamma}_0\boldsymbol{\Pi}_0' + \mathbf{B}_0 = \mathbf{0}$. By the same argument given in (e) with $\boldsymbol{\delta}_m$ replaced by $\boldsymbol{\delta}_{0m}$ shows that $\boldsymbol{\delta}_{0m}$ is *a* solution to (10). Rewrite (10) by taking the transpose:

$$\mathbf{A}\mathbf{x} = \mathbf{y} \text{ with } \mathbf{A} \equiv [\boldsymbol{\Pi}_0\mathbf{S}_m' \vdots \mathbf{C}_m'], \mathbf{x} \equiv \boldsymbol{\delta}_m, \mathbf{y} \equiv \boldsymbol{\pi}_{0m}. \tag{10$'$}$$

A necessary and sufficient condition that $\boldsymbol{\delta}_{0m}$ is the only solution to (10$'$) is that the coefficient matrix in (10$'$), which is $K \times L_m$ (where $L_m = M_m + K_m$), be of full column rank (that is, the rank of the matrix be equal to the number of columns, which is L_m). We have shown in (a) that this condition is equivalent to the rank condition for identification for the m-th equation.

(g) The hint is the answer.

4. In this part, we let $\mathbf{F}_m$ stand for the $K \times L_m$ matrix $[\boldsymbol{\Pi}_0\mathbf{S}_m' \vdots \mathbf{C}_m']$. Since x_{tK} does not appear in the system, the last row of $\boldsymbol{\Pi}_0$ is a vector of zeros and the last row of $\mathbf{C}_m'$ is a vector of zeros. So the last row of of $\mathbf{F}_m$ is a vector of zeros:

$$\mathbf{F}_m = \begin{bmatrix} \underset{((K-1)\times L_m)}{\widetilde{\mathbf{F}}_m} \\ \underset{(1\times L_m)}{\mathbf{0}'} \end{bmatrix}.$$

Dropping x_{tK} from the list of instruments means dropping the last row of $\mathbf{F}_m$, which does not alter the full column rank condition. The asymptotic variance of the FIML estimator is given in (4.5.15) with (4.5.16) on p. 278. Using (6) on (4.5.16), we obtain

$$\mathbf{A}_{mh} = \mathbf{F}_m' \operatorname{E}(\mathbf{x}_t\mathbf{x}_t')\mathbf{F}_h = \begin{bmatrix} \widetilde{\mathbf{F}}_m' & \mathbf{0} \end{bmatrix} \begin{bmatrix} \operatorname{E}(\widetilde{\mathbf{x}}_t\widetilde{\mathbf{x}}_t') & \operatorname{E}(x_{tK}\widetilde{\mathbf{x}}_t) \\ \operatorname{E}(x_{tK}\widetilde{\mathbf{x}}_t') & \operatorname{E}(x_{tK}^2) \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{F}}_h \\ \mathbf{0}' \end{bmatrix} = \widetilde{\mathbf{F}}_m' \operatorname{E}(\widetilde{\mathbf{x}}_t\widetilde{\mathbf{x}}_t')\widetilde{\mathbf{F}}_h.$$

This shows that the asymptotic variance is unchanged when x_{tK} is dropped.