Solution to Chapter 6 Analytical Exercises

1. The hint is the answer.

2. (a) Let $\sigma_n \equiv \sum_{j=0}^{n} \psi_j^2$. Then

$$\begin{aligned}
\mathrm{E}[(y_{t,m} - y_{t,n})^2] &= \mathrm{E}\left[\Big(\sum_{j=n+1}^{m} \psi_j \varepsilon_{t-j}\Big)^2\right] \\
&= \sigma^2 \sum_{j=n+1}^{m} \psi_j^2 \qquad \text{(since } \{\varepsilon_t\} \text{ is white noise)} \\
&= \sigma^2 \, |\alpha_m - \alpha_n|.
\end{aligned}$$

Since $\{\psi_j\}$ is absolutely summable (and hence square summable), $\{\alpha_n\}$ converges. So $|\alpha_m - \alpha_n| \to \infty$ as $m, n \to \infty$. Therefore, $\mathrm{E}[(y_{t,m} - y_{t,n})^2] \to 0$ as $m, n \to \infty$, which means $\{y_{t,n}\}$ converges in mean square in n by (i).

(b) Since $y_{t,n} \to_{\text{m.s.}} y_t$ as shown in (a), $\mathrm{E}(y_t) = \lim_{n\to\infty} \mathrm{E}(y_{t,n})$ by (ii). But $\mathrm{E}(y_{t,n}) = 0$.

(c) Since $y_{t,n} - \mu \to_{\text{m.s.}} y_t - \mu$ and $y_{t-j,n} - \mu \to_{\text{m.s.}} y_{t-j} - \mu$ as $n \to \infty$,

$$\mathrm{E}[(y_t - \mu)(y_{t-j} - \mu)] = \lim_{n\to\infty} \mathrm{E}[(y_{t,n} - \mu)(y_{t-j,n} - \mu)].$$

(d) (reproducing the answer on pp. 441-442 of the book) Since $\{\psi_j\}$ is absolutely summable, $\psi_j \to 0$ as $j \to \infty$. So for any j, there exists an $A > 0$ such that $|\psi_{j+k}| \le A$ for all j, k. So $|\psi_{j+k} \cdot \psi_k| \le A|\psi_k|$. Since $\{\psi_k\}$ (and hence $\{A\psi_k\}$) is absolutely summable, so is $\{\psi_{j+k} \cdot \psi_k\}$ $(k = 0, 1, 2, \ldots)$ for any given j. Thus by (i),

$$|\gamma_j| = \sigma^2 \left|\sum_{k=0}^{\infty} \psi_{j+k}\psi_k\right| \le \sigma^2 \sum_{k=0}^{\infty} |\psi_{j+k}\psi_k| = \sigma^2 \sum_{k=0}^{\infty} |\psi_{j+k}|\,|\psi_k| < \infty.$$

Now set a_{jk} in (ii) to $|\psi_{j+k}| \cdot |\psi_k|$. Then

$$\sum_{j=0}^{\infty} |a_{jk}| = \sum_{j=0}^{\infty} |\psi_k|\,|\psi_{j+k}| \le |\psi_k| \sum_{j=0}^{\infty} |\psi_j| < \infty.$$

Let

$$M \equiv \sum_{j=0}^{\infty} |\psi_j| \quad \text{and} \quad s_k \equiv |\psi_k| \sum_{j=0}^{\infty} |\psi_{j+k}|.$$

Then $\{s_k\}$ is summable because $|s_k| \le |\psi_k| \cdot M$ and $\{\psi_k\}$ is absolutely summable. Therefore, by (ii),

$$\sum_{j=0}^{\infty}\Big(\sum_{k=0}^{\infty} |\psi_{j+k}| \cdot |\psi_k|\Big) < \infty.$$

This and the first inequality above mean that $\{\gamma_j\}$ is absolutely summable.

3. (a)

$$\begin{aligned}\gamma_j &= \mathrm{Cov}(y_{t,n}, y_{t-j,n}) \\ &= \mathrm{Cov}(h_0 x_t + h_1 x_{t-1} + \cdots + h_n x_{t-n}, h_0 x_{t-j} + h_1 x_{t-j-1} + \cdots + h_n x_{t-j-n}) \\ &= \sum_{k=0}^{n}\sum_{\ell=0}^{n} h_k h_\ell \,\mathrm{Cov}(x_{t-k}, x_{t-j-\ell}) \\ &= \sum_{k=0}^{n}\sum_{\ell=0}^{n} h_k h_\ell \,\gamma^x_{j+\ell-k}.\end{aligned}$$

(b) Since $\{h_j\}$ is absolutely summable, we have $y_{t,n} \to_{\text{m.s.}} y_t$ as $n \to \infty$ by Proposition 6.2(a). Then, using the facts (i) and (ii) displayed in Analytical Exercise 2, we can show:

$$\begin{aligned}&\sum_{k=0}^{n}\sum_{\ell=0}^{n} h_k h_\ell \,\gamma^x_{j+\ell-k} = \mathrm{Cov}(y_{t,n}, y_{t-j,n}) \\ &= \mathrm{E}(y_{t,n}\, y_{t-j,n}) - \mathrm{E}(y_{t,n})\,\mathrm{E}(y_{t-j,n}) \to \mathrm{E}(y_t\, y_{t-j}) - \mathrm{E}(y_t)\,\mathrm{E}(y_{t-j}) = \mathrm{Cov}(y_t, y_{t-j})\end{aligned}$$

as $n \to \infty$. That is, $\sum_{k=0}^{n}\sum_{\ell=0}^{n} h_k h_\ell \,\gamma^x_{j+\ell-k}$ converges as $n \to \infty$, which is the desired result.

4. (a) (8) solves the difference equation $y_j - \phi_1 y_{j-1} - \phi_2 y_{j-2} = 0$ because

$$\begin{aligned}&y_j - \phi_1 y_{j-1} - \phi_2 y_{j-2} \\ &= (c_{10}\lambda_1^{-j} + c_{20}\lambda_2^{-j}) - \phi_1(c_{10}\lambda_1^{-j+1} + c_{20}\lambda_2^{-j+1}) - \phi_2(c_{10}\lambda_1^{-j+2} + c_{20}\lambda_2^{-j+2}) \\ &= c_{10}\lambda_1^{-j}(1 - \phi_1\lambda_1 - \phi_2\lambda_1^2) + c_{20}\lambda_1^{-j}(1 - \phi_1\lambda_2 - \phi_2\lambda_2^2) \\ &= 0 \qquad (\text{since } \lambda_1 \text{ and } \lambda_2 \text{ are the roots of } 1 - \phi_1 z - \phi_2 z^2 = 0).\end{aligned}$$

Writing down (8) for $j = 0, 1$ gives

$$y_0 = c_{10} + c_{20},\ y_1 = c_{10}\lambda_1^{-1} + c_{20}\lambda_2^{-1}.$$

Solve this for (c_{10}, c_{20}) given $(y_0, y_1, \lambda_1, \lambda_2)$.

(b) This should be easy.

(c) For $j \geq J$, we have $j^n \xi^j < b^j$. Define B as

$$B \equiv \max\left\{\frac{\xi}{b}, \frac{2^n \xi^j}{b^2}, \frac{3^n \xi^3}{b^3}, \ldots, \frac{(J-1)^n \xi^{J-1}}{b^{J-1}}\right\}.$$

Then, by construction,

$$B \geq \frac{j^n \xi^j}{b^j} \quad \text{or} \quad j^n \xi^j \leq B\, b^j$$

for $j = 0, 1, .., J-1$. Choose A so that $A > 1$ and $A > B$. Then $j^n \xi^j < b^j < A\, b^j$ for $j \geq J$ and $j^n \xi^j \leq B\, b^j < A\, b^j$ for all $j = 0, 1, \ldots, J-1$.

(d) The hint is the answer.

5. (a) Multiply both sides of (6.2.1′) by $y_{t-j} - \mu$ and take the expectation of both sides to derive the desired result.

(b) The result follows immediately from the MA representation $y_{t-j} - \mu = \varepsilon_{t-j} + \phi\, \varepsilon_{t-j-1} + \phi^2\, \varepsilon_{t-j-2} + \cdots$.

(c) Immediate from (a) and (b).

(d) Set $j = 1$ in (10) to obtain $\gamma_1 - \rho\gamma_0 = 0$. Combine this with (9) to solve for (γ_0, γ_1):

$$\gamma_0 = \frac{\sigma^2}{1-\phi^2}, \quad \gamma_1 = \frac{\sigma^2}{1-\phi^2}\phi.$$

Then use (10) as the first-order difference equation for $j = 2, 3, \ldots$ in γ_j with the initial condition $\gamma_1 = \frac{\sigma^2}{1-\phi^2}\phi$. This gives: $\gamma_j = \frac{\sigma^2}{1-\phi^2}\phi^j$, verifying (6.2.5).

6. (a) Should be obvious.

(b) By the definition of mean-square convergence, what needs to be shown is that $\mathrm{E}[(x_t - x_{t,n})^2] \to 0$ as $n \to \infty$.

$$\begin{aligned} \mathrm{E}[(x_t - x_{t,n})^2] &= \mathrm{E}[(\phi^n x_{t-n})^2] \qquad (\text{since } x_t = x_{t,n} + \phi^n\, x_{t-n}) \\ &= \phi^{2n}\ \mathrm{E}(x_{t-n}^2) \\ &\to 0 \qquad (\text{since } |\phi| < 1 \text{ and } \mathrm{E}(x_{t-n}^2) < \infty). \end{aligned}$$

(c) Should be obvious.

7. (d) By the hint, what needs to be shown is that $(\mathbf{F})^n \boldsymbol{\xi}_{t-n} \to_{\text{m.s.}} \mathbf{0}$. Let $\mathbf{z}_n \equiv (\mathbf{F})^n \boldsymbol{\xi}_{t-n}$. Contrary to the suggestion of the hint, which is to show the mean-square convergence of the components of $\mathbf{z}_n$, here we show an equivalent claim (see Review Question 2 to Section 2.1) that $\lim_{n\to\infty} \mathrm{E}(\mathbf{z}_n'\mathbf{z}_n) = 0$.

$$\mathbf{z}_n'\mathbf{z}_n = \mathrm{trace}(\mathbf{z}_n'\mathbf{z}_n) = \mathrm{trace}[\boldsymbol{\xi}_{t-n}'[(\mathbf{F})^n]'[(\mathbf{F})^n]\boldsymbol{\xi}_{t-n}] = \mathrm{trace}\{\boldsymbol{\xi}_{t-n}\boldsymbol{\xi}_{t-n}'[(\mathbf{F})^n]'[(\mathbf{F})^n]\}$$

Since the trace and the expectations operator can be interchanged,

$$\mathrm{E}(\mathbf{z}_n'\mathbf{z}_n) = \mathrm{trace}\{\mathrm{E}(\boldsymbol{\xi}_{t-n}\boldsymbol{\xi}_{t-n}')[(\mathbf{F})^n]'[(\mathbf{F})^n]\}.$$

Since $\boldsymbol{\xi}_t$ is covariance-stationary, we have $\mathrm{E}(\boldsymbol{\xi}_{t-n}\boldsymbol{\xi}_{t-n}') = \mathbf{V}$ (the autocovariance matrix). Since all the roots of the characteristic equation are less than one in absolute value, $\mathbf{F}^n = \mathbf{T}(\boldsymbol{\Lambda})^n\mathbf{T}^{-1}$ converges to a zero matrix. We can therefore conclude that $\mathrm{E}(\mathbf{z}_n'\mathbf{z}_n) \to 0$.

(e) ψ_n is the (1,1) element of $\mathbf{T}(\boldsymbol{\Lambda})^n\mathbf{T}^{-1}$.

8. (a)

$$\begin{aligned} \mathrm{E}(y_t) &= \frac{1-\phi^t}{1-\phi}c + \phi^t\,\mathrm{E}(y_0) \to \frac{c}{1-\phi}, \\ \mathrm{Var}(y_t) &= \frac{1-\phi^{2t}}{1-\phi^2}\sigma^2 + \phi^{2t}\,\mathrm{Var}(y_0) \to \frac{\sigma^2}{1-\phi^2}, \\ \mathrm{Cov}(y_t, y_{t-j}) &= \phi^j\left[\frac{1-\phi^{2(t-j)}}{1-\phi^2}\sigma^2 + \phi^{2(t-j)}\,\mathrm{Var}(y_0)\right] \to \phi^j\frac{\sigma^2}{1-\phi^2}. \end{aligned}$$

(b) This should be easy to verify given the above formulas.

9. (a) The hint is the answer.

(b) Since $\gamma_j \to 0$, the result proved in (a) implies that $\frac{2}{n}\sum_{j=1}^{n}|\gamma_j| \to 0$. Also, $\gamma_0/n \to 0$. So by the inequality for $\mathrm{Var}(\overline{y})$ shown in the question, $\mathrm{Var}(\overline{y}) \to 0$.

10. (a) By the hint,

$$\sum_{j=1}^{n} j\,a_j \leq \sum_{j=1}^{N} \left| \sum_{k=j}^{n} a_k \right| + \sum_{j=N+1}^{n} \left| \sum_{k=j}^{n} a_k \right| < NM + (n-N)\frac{\varepsilon}{2}.$$

So

$$\frac{1}{n}\sum_{j=1}^{n} j\,a_j < \frac{NM}{n} + \frac{n-N}{n}\frac{\varepsilon}{2} < \frac{NM}{n} + \frac{\varepsilon}{2}.$$

By taking n large enough, NM/n can be made less than $\varepsilon/2$.

(b) From (6.5.2),

$$\mathrm{Var}(\sqrt{n}\,\overline{y}) = \gamma_0 + 2\sum_{j=1}^{n-1}\left(1-\frac{j}{n}\right)\gamma_j = \left[\gamma_0 + 2\sum_{j=1}^{n-1}\gamma_j\right] - \frac{2}{n}\sum_{j=1}^{n-1} j\,\gamma_j.$$

The term in brackets converges to $\sum_{j=-\infty}^{\infty}\gamma_j$ if $\{\gamma_j\}$ is summable. (a) has shown that the last term converges to zero if $\{\gamma_j\}$ is summable.