Solution to Chapter 5 Analytical Exercises

1. (a) Let $(\mathbf{a}', \mathbf{b}')'$ be the OLS estimate of $(\boldsymbol{\alpha}', \boldsymbol{\beta}')'$. Define $\mathbf{M_D}$ as in equation (4) of the hint. By the Frisch-Waugh theorem, $\mathbf{b}$ is the OLS coefficient estimate in the regression of $\mathbf{M_D y}$ on $\mathbf{M_D F}$. The proof is complete if we can show the claim that

$$\widetilde{\mathbf{y}} = \mathbf{M_D y} \quad \text{and} \quad \widetilde{\mathbf{F}} = \mathbf{M_D F},$$

where $\widetilde{\mathbf{y}}$ and $\widetilde{\mathbf{F}}$ are defined in (5.2.2) and (5.2.3). This is because the fixed-effects estimator can be written as $(\widetilde{\mathbf{F}}'\widetilde{\mathbf{F}})^1\,\widetilde{\mathbf{F}}'\widetilde{\mathbf{y}}$ (see (5.2.4)). But the above claim follows immediately if we can show that $\mathbf{M_D} = \mathbf{I}_n \otimes \mathbf{Q}$, where $\mathbf{Q} \equiv \mathbf{I}_M - \frac{1}{M}\mathbf{1}_M\mathbf{1}_M'$, the annihilator associated with $\mathbf{1}_M$.

$$\begin{aligned}
\mathbf{M_D} &= \mathbf{I}_{Mn} - (\mathbf{I}_n \otimes \mathbf{1}_M)\left[(\mathbf{I}_n \otimes \mathbf{1}_M)'(\mathbf{I}_n \otimes \mathbf{1}_M)\right]^{-1}(\mathbf{I}_n \otimes \mathbf{1}_M)' \\
&= \mathbf{I}_{Mn} - (\mathbf{I}_n \otimes \mathbf{1}_M)\left[(\mathbf{I}_n \otimes \mathbf{1}_M'\mathbf{1}_M)\right]^{-1}(\mathbf{I}_n \otimes \mathbf{1}_M') \\
&= \mathbf{I}_{Mn} - (\mathbf{I}_n \otimes \mathbf{1}_M)\left[(\mathbf{I}_n \otimes M)\right]^{-1}(\mathbf{I}_n \otimes \mathbf{1}_M') \\
&= \mathbf{I}_{Mn} - (\mathbf{I}_n \otimes \mathbf{1}_M)(\mathbf{I}_n \otimes \frac{1}{M})(\mathbf{I}_n \otimes \mathbf{1}_M') \\
&= \mathbf{I}_{Mn} - (\mathbf{I}_n \otimes \frac{1}{M}\mathbf{1}_M\mathbf{1}_M') \\
&= (\mathbf{I}_n \otimes \mathbf{I}_M) - (\mathbf{I}_n \otimes \frac{1}{M}\mathbf{1}_M\mathbf{1}_M') \\
&= (\mathbf{I}_n \otimes (\mathbf{I}_M - \frac{1}{M}\mathbf{1}_M\mathbf{1}_M')) \\
&= \mathbf{I}_n \otimes \mathbf{Q}.
\end{aligned}$$

(b) As indicated in the hint to (a), we have $\mathbf{a} = (\mathbf{D}'\mathbf{D})^{-1}(\mathbf{D}'\mathbf{y} - \mathbf{D}'\mathbf{F}\mathbf{b})$. It should be straightforward to show that

$$\mathbf{D}'\mathbf{D} = M\,\mathbf{I}_n, \quad \mathbf{D}'\mathbf{y} = \begin{bmatrix} \mathbf{1}_M'\mathbf{y}_1 \\ \vdots \\ \mathbf{1}_M'\mathbf{y}_n \end{bmatrix}, \quad \mathbf{D}'\mathbf{F}\mathbf{b} = \begin{bmatrix} \mathbf{1}_M'\mathbf{F}_1\mathbf{b} \\ \vdots \\ \mathbf{1}_M'\mathbf{F}_n\mathbf{b} \end{bmatrix}.$$

Therefore,

$$\mathbf{a} = \begin{bmatrix} \frac{1}{M}(\mathbf{1}_M'\mathbf{y}_1 - \mathbf{1}_M'\mathbf{F}_1\mathbf{b}) \\ \vdots \\ \frac{1}{M}(\mathbf{1}_M'\mathbf{y}_n - \mathbf{1}_M'\mathbf{F}_n\mathbf{b}) \end{bmatrix}.$$

The desired result follows from this because $\mathbf{b}$ equals the fixed-effects estimator $\widehat{\boldsymbol{\beta}}_{\text{FE}}$ and

$$\mathbf{1}_M'\mathbf{y}_i = (y_{i1} + \cdots + y_{iM}) \quad \text{and} \quad \mathbf{1}_M'\mathbf{F}_n\mathbf{b} = \mathbf{1}_M' \begin{bmatrix} \mathbf{f}_{i1}' \\ \vdots \\ \mathbf{f}_{iM}' \end{bmatrix} \mathbf{b} = \left(\sum_{m=1}^{M} \mathbf{f}_{im}'\right)\mathbf{b}.$$

(c) What needs to be shown is that (3) and conditions (i)-(iv) listed in the question together imply Assumptions 1.1-1.4. Assumption 1.1 (linearity) is none other than (3). Assumption 1.3 is a restatement of (iv). This leaves Assumptions 1.2 (strict exogeneity) and Assumption 1.4 (spherical error term) to be verified. The following is an amplification of the answer to 1.(c) on p. 363.

$$\begin{aligned}
\mathrm{E}(\boldsymbol{\eta}_i \mid \mathbf{W}) &= \mathrm{E}(\boldsymbol{\eta}_i \mid \mathbf{F}) \quad \text{(since } \mathbf{D} \text{ is a matrix of constants)} \\
&= \mathrm{E}(\boldsymbol{\eta}_i \mid \mathbf{F}_1, \ldots, \mathbf{F}_n) \\
&= \mathrm{E}(\boldsymbol{\eta}_i \mid \mathbf{F}_i) \qquad \text{(since } (\boldsymbol{\eta}_i, \mathbf{F}_i) \text{ is indep. of } \mathbf{F}_j \text{ for } j \neq i) \text{ by (i)} \\
&= \mathbf{0} \qquad \text{(by (ii))}.
\end{aligned}$$

Therefore, the regressors are strictly exogenous (Assumption 1.2). Also,

$$\begin{aligned}
\mathrm{E}(\boldsymbol{\eta}_i \boldsymbol{\eta}_i' \mid \mathbf{W}) &= \mathrm{E}(\boldsymbol{\eta}_i \boldsymbol{\eta}_i' \mid \mathbf{F}) \\
&= \mathrm{E}(\boldsymbol{\eta}_i \boldsymbol{\eta}_i' \mid \mathbf{F}_i) \\
&= \sigma_\eta^2 \mathbf{I}_M \quad \text{(by the spherical error assumption (iii))}.
\end{aligned}$$

For $i \neq j$,

$$\begin{aligned}
\mathrm{E}(\boldsymbol{\eta}_i \boldsymbol{\eta}_j' \mid \mathbf{W}) &= \mathrm{E}(\boldsymbol{\eta}_i \boldsymbol{\eta}_j' \mid \mathbf{F}) \\
&= \mathrm{E}(\boldsymbol{\eta}_i \boldsymbol{\eta}_j' \mid \mathbf{F}_1, \ldots, \mathbf{F}_n) \\
&= \mathrm{E}(\boldsymbol{\eta}_i \boldsymbol{\eta}_j' \mid \mathbf{F}_i, \mathbf{F}_j) \qquad \text{(since } (\boldsymbol{\eta}_i, \mathbf{F}_i, \boldsymbol{\eta}_j, \mathbf{F}_j) \text{ is indep. of } \mathbf{F}_k \text{ for } k \neq i, j \text{ by (i))} \\
&= \mathrm{E}[\mathrm{E}(\boldsymbol{\eta}_i \boldsymbol{\eta}_j' \mid \mathbf{F}_i, \mathbf{F}_j, \boldsymbol{\eta}_i) \mid \mathbf{F}_i, \mathbf{F}_j] \\
&= \mathrm{E}[\boldsymbol{\eta}_i \, \mathrm{E}(\boldsymbol{\eta}_j' \mid \mathbf{F}_i, \mathbf{F}_j, \boldsymbol{\eta}_i) \mid \mathbf{F}_i, \mathbf{F}_j] \\
&= \mathrm{E}[\boldsymbol{\eta}_i \, \mathrm{E}(\boldsymbol{\eta}_j' \mid \mathbf{F}_j) \mid \mathbf{F}_i, \mathbf{F}_j] \qquad \text{(since } (\boldsymbol{\eta}_j, \mathbf{F}_j) \text{ is independent of } (\boldsymbol{\eta}_i, \mathbf{F}_i) \text{ by (i))} \\
&= \mathbf{0} \qquad \text{(since } \mathrm{E}(\boldsymbol{\eta}_j' \mid \mathbf{F}_j) \text{ by (ii))}.
\end{aligned}$$

So $\mathrm{E}(\boldsymbol{\eta}\boldsymbol{\eta}' \mid \mathbf{W}) = \sigma_\eta^2 \mathbf{I}_{Mn}$ (Assumption 1.4).

Since the assumptions of the classical regression model are satisfied, Propositions 1.1 holds for the OLS estimator $(\mathbf{a}, \mathbf{b})$. The estimator is unbiased and the Gauss-Markov theorem holds.

As shown in Analytical Exercise 4.(f) in Chapter 1, the residual vector from the original regression (3) (which is to regress $\mathbf{y}$ on $\mathbf{D}$ and $\mathbf{F}$) is numerically the same as the residual vector from the regression of $\widetilde{\mathbf{y}}$ $(= \mathbf{M_D y})$ on $\widetilde{\mathbf{F}}$ $(= \mathbf{M_D F})$). So the two *SSR*'s are the same.

2. (a) It is evident that $\mathbf{C}'\mathbf{1}_M = \mathbf{0}$ if $\mathbf{C}$ is what is referred to in the question as the matrix of first differences. Next, to see that $\mathbf{C}'\mathbf{1}_M = \mathbf{0}$ if $\mathbf{C}$ is an $M \times (M-1)$ matrix created by dropping one column from $\mathbf{Q}$, first note that by construction of $\mathbf{Q}$, we have:

$$\underset{(M\times M)}{\mathbf{Q}} \mathbf{1}_M = \underset{(M\times 1)}{\mathbf{0}},$$

which is a set of M equations. Drop one row from $\mathbf{Q}$ and call it $\mathbf{C}'$ and drop the corresponding element from the $\mathbf{0}$ vector on the RHS. Then

$$\underset{((M-1)\times M)}{\mathbf{C}'} \mathbf{1}_M = \underset{((M-1)\times 1)}{\mathbf{0}}.$$

(b) By multiplying both sides of (5.1.1″) on p. 329 by $\mathbf{C}'$, we eliminate $\mathbf{1}_M \cdot \mathbf{b}_i \boldsymbol{\gamma}$ and $\mathbf{1}_M \cdot \alpha_i$.

(c) Below we verify the five conditions.

- The random sample condition is immediate from (5.1.2).
- Regarding the orthogonality conditions, as mentioned in the hint, (5.1.8b) can be written as $\mathrm{E}(\boldsymbol{\eta}_i \otimes \mathbf{x}_i) = \mathbf{0}$. This implies the orthogonality conditions because

$$\mathrm{E}(\widehat{\boldsymbol{\eta}}_i \otimes \mathbf{x}_i) = \mathrm{E}[(\mathbf{C}' \otimes \mathbf{I}_K)(\boldsymbol{\eta}_i \otimes \mathbf{x}_i)] = (\mathbf{C}' \otimes \mathbf{I}_K)\,\mathrm{E}(\boldsymbol{\eta}_i \otimes \mathbf{x}_i).$$

- As shown on pp. 363-364, the identification condition to be verified is equivalent to (5.1.15) (that $\mathrm{E}(\mathbf{QF}_i \otimes \mathbf{x}_i)$ be of full column rank).
- Since $\boldsymbol{\varepsilon}_i = \mathbf{1}_M \cdot \alpha_i + \boldsymbol{\eta}_i$, we have $\widehat{\boldsymbol{\eta}}_i \equiv \mathbf{C}'\boldsymbol{\eta}_i = \mathbf{C}'\boldsymbol{\varepsilon}_i$. So $\widehat{\boldsymbol{\eta}}_i\widehat{\boldsymbol{\eta}}_i' = \mathbf{C}'\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i'\mathbf{C}$ and

$$\mathrm{E}(\widehat{\boldsymbol{\eta}}_i\widehat{\boldsymbol{\eta}}_i' \mid \mathbf{x}_i) = \mathrm{E}(\mathbf{C}'\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i'\mathbf{C} \mid \mathbf{x}_i) = \mathbf{C}'\,\mathrm{E}(\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i' \mid \mathbf{x}_i)\mathbf{C} = \mathbf{C}'\boldsymbol{\Sigma}\mathbf{C}.$$

  (The last equality is by (5.1.5).)
- By the definition of $\widehat{\mathbf{g}}_i$, we have: $\widehat{\mathbf{g}}_i\widehat{\mathbf{g}}_i' = \widehat{\boldsymbol{\eta}}_i\widehat{\boldsymbol{\eta}}_i' \otimes \mathbf{x}_i\mathbf{x}_i'$. But as just shown above, $\widehat{\boldsymbol{\eta}}_i\widehat{\boldsymbol{\eta}}_i' = \mathbf{C}'\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i'\mathbf{C}$. So

$$\widehat{\mathbf{g}}_i\widehat{\mathbf{g}}_i' = \mathbf{C}'\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i'\mathbf{C} \otimes \mathbf{x}_i\mathbf{x}_i' = (\mathbf{C}' \otimes \mathbf{I}_K)(\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i' \otimes \mathbf{x}_i\mathbf{x}_i')(\mathbf{C} \otimes \mathbf{I}_K).$$

  Thus

$$\begin{aligned}\mathrm{E}(\widehat{\mathbf{g}}_i\widehat{\mathbf{g}}_i') &= (\mathbf{C}' \otimes \mathbf{I}_K)\,\mathrm{E}[(\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i' \otimes \mathbf{x}_i\mathbf{x}_i')](\mathbf{C} \otimes \mathbf{I}_K) \\ &= (\mathbf{C}' \otimes \mathbf{I}_K)\,\mathrm{E}(\mathbf{g}_i\mathbf{g}_i')(\mathbf{C} \otimes \mathbf{I}_K) \qquad (\text{since } \mathbf{g}_i \equiv \boldsymbol{\varepsilon}_i \otimes \mathbf{x}_i).\end{aligned}$$

  Since $\mathrm{E}(\mathbf{g}_i\mathbf{g}_i')$ is non-singular by (5.1.6) and since $\mathbf{C}$ is of full column rank, $\mathrm{E}(\widehat{\mathbf{g}}_i\widehat{\mathbf{g}}_i')$ is non-singular.

(d) Since $\widehat{\mathbf{F}}_i \equiv \mathbf{C}'\mathbf{F}_i$, we can rewrite $\mathbf{S}_{\mathbf{xz}}$ and $\mathbf{s}_{\mathbf{xy}}$ as

$$\mathbf{S}_{\mathbf{xz}} = (\mathbf{C}' \otimes \mathbf{I}_K)\Big(\frac{1}{n}\sum_{i=1}^n \mathbf{F}_i \otimes \mathbf{x}_i\Big),\ \mathbf{s}_{\mathbf{xy}} = (\mathbf{C}' \otimes \mathbf{I}_K)\Big(\frac{1}{n}\sum_{i=1}^n \mathbf{y}_i \otimes \mathbf{x}_i\Big).$$

So

$$\begin{aligned}\mathbf{S}_{\mathbf{xz}}'\widehat{\mathbf{W}}\mathbf{S}_{\mathbf{xz}} &= \Big(\frac{1}{n}\sum_{i=1}^n \mathbf{F}_i' \otimes \mathbf{x}_i'\Big)(\mathbf{C} \otimes \mathbf{I}_K)\left[(\mathbf{C}'\mathbf{C})^{-1} \otimes \Big(\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i'\Big)^{-1}\right](\mathbf{C}' \otimes \mathbf{I}_K)\Big(\frac{1}{n}\sum_{i=1}^n \mathbf{F}_i \otimes \mathbf{x}_i\Big) \\ &= \Big(\frac{1}{n}\sum_{i=1}^n \mathbf{F}_i' \otimes \mathbf{x}_i'\Big)\left[\mathbf{C}(\mathbf{C}'\mathbf{C})^{-1}\mathbf{C}' \otimes \Big(\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i'\Big)^{-1}\right]\Big(\frac{1}{n}\sum_{i=1}^n \mathbf{F}_i \otimes \mathbf{x}_i\Big) \\ &= \Big(\frac{1}{n}\sum_{i=1}^n \mathbf{F}_i' \otimes \mathbf{x}_i'\Big)\left[\mathbf{Q} \otimes \Big(\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i'\Big)^{-1}\right]\Big(\frac{1}{n}\sum_{i=1}^n \mathbf{F}_i \otimes \mathbf{x}_i\Big) \\ &\qquad (\text{since } \mathbf{C}(\mathbf{C}'\mathbf{C})^{-1}\mathbf{C}' = \mathbf{Q}, \text{ as mentioned in the hint}).\end{aligned}$$

Similarly,

$$\mathbf{S}_{\mathbf{xz}}'\widehat{\mathbf{W}}\mathbf{s}_{\mathbf{xy}} = \Big(\frac{1}{n}\sum_{i=1}^n \mathbf{F}_i' \otimes \mathbf{x}_i'\Big)\left[\mathbf{Q} \otimes \Big(\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i'\Big)^{-1}\right]\Big(\frac{1}{n}\sum_{i=1}^n \mathbf{y}_i \otimes \mathbf{x}_i\Big).$$

Noting that $\mathbf{f}'_{im}$ is the m-th row of $\mathbf{F}_i$ and writing out the Kronecker products in full, we obtain

$$\mathbf{S}'_{\mathbf{xz}}\widehat{\mathbf{W}}\mathbf{S}_{\mathbf{xz}} = \sum_{m=1}^{M}\sum_{h=1}^{M} q_{mh}\Big\{\Big(\frac{1}{n}\sum_{i=1}^{n}\mathbf{f}_{im}\mathbf{x}'_i\Big)\Big(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}'_i\Big)^{-1}\Big(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{f}'_{ih}\Big)\Big\},$$

$$\mathbf{S}'_{\mathbf{xz}}\widehat{\mathbf{W}}\mathbf{s}_{\mathbf{xy}} = \sum_{m=1}^{M}\sum_{h=1}^{M} q_{mh}\Big\{\Big(\frac{1}{n}\sum_{i=1}^{n}\mathbf{f}_{im}\mathbf{x}'_i\Big)\Big(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}'_i\Big)^{-1}\Big(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\cdot y_{ih}\Big)\Big\},$$

where q_{mh} is the (m,h) element of $\mathbf{Q}$. (This is just (4.6.6) with $\mathbf{x}_{im} = \mathbf{x}_i$, $\mathbf{z}_{im} = \mathbf{f}_{im}$, $\widehat{\mathbf{W}} = \mathbf{Q}\otimes\Big(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}'_i\Big)^{-1}$.) Since $\mathbf{x}_i$ includes all the elements of $\mathbf{F}_i$, as noted in the hint, $\mathbf{x}_i$ "dissappears". So

$$\mathbf{S}'_{\mathbf{xz}}\widehat{\mathbf{W}}\mathbf{S}_{\mathbf{xz}} = \sum_{m=1}^{M}\sum_{h=1}^{M} q_{mh}\Big(\frac{1}{n}\sum_{i=1}^{n}\mathbf{f}_{im}\mathbf{f}'_{ih}\Big) = \frac{1}{n}\sum_{i=1}^{n}\Big(\sum_{m=1}^{M}\sum_{h=1}^{M} q_{mh}\mathbf{f}_{im}\mathbf{f}'_{ih}\Big),$$

$$\mathbf{S}'_{\mathbf{xz}}\widehat{\mathbf{W}}\mathbf{s}_{\mathbf{xy}} = \sum_{m=1}^{M}\sum_{h=1}^{M} q_{mh}\frac{1}{n}\sum_{i=1}^{n}\mathbf{f}_{im}\cdot y_{ih} = \frac{1}{n}\sum_{i=1}^{n}\Big(\sum_{m=1}^{M}\sum_{h=1}^{M} q_{mh}\mathbf{f}_{im}\cdot y_{ih}\Big).$$

Using the "beautifying" formula (4.6.16b), this expression can be simplified as

$$\mathbf{S}'_{\mathbf{xz}}\widehat{\mathbf{W}}\mathbf{S}_{\mathbf{xz}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{F}'_i\mathbf{Q}\mathbf{F}_i,$$

$$\mathbf{S}'_{\mathbf{xz}}\widehat{\mathbf{W}}\mathbf{s}_{\mathbf{xy}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{F}'_i\mathbf{Q}\mathbf{y}_i.$$

So $\Big(\mathbf{S}'_{\mathbf{xz}}\widehat{\mathbf{W}}\mathbf{S}_{\mathbf{xz}}\Big)^{-1}\mathbf{S}'_{\mathbf{xz}}\widehat{\mathbf{W}}\mathbf{s}_{\mathbf{xy}}$ is the fixed-effects estimator.

(e) The previous part shows that the fixed-effects estimator is not efficient because the $\widehat{\mathbf{W}}$ in (10) does not satisfy the efficiency condition that $\text{plim}\,\widehat{\mathbf{W}} = \mathbf{S}^{-1}$. Under conditional homoskedasticity, $\mathbf{S} = \text{E}(\widehat{\boldsymbol{\eta}}_i\widehat{\boldsymbol{\eta}}'_i)\otimes\text{E}(\mathbf{x}_i\mathbf{x}'_i)$. Thus, with $\widehat{\boldsymbol{\Psi}}$ being a consistent estimator of $\text{E}(\widehat{\boldsymbol{\eta}}_i\widehat{\boldsymbol{\eta}}'_i)$, the efficient GMM estimator is given by setting

$$\widehat{\mathbf{W}} = \widehat{\boldsymbol{\Psi}}^{-1}\otimes\Big(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}'_i\Big)^{-1}.$$

This is none other than the random-effects estimator applied to the system of $M-1$ equations (9). By setting $\mathbf{Z}_i = \widehat{\mathbf{F}}_i$, $\widehat{\boldsymbol{\Sigma}} = \widehat{\boldsymbol{\Psi}}$, $\mathbf{y}_i = \widehat{\mathbf{y}}_i$ in (4.6.8′) and (4.6.9′) on p. 293, we obtain (12) and (13) in the question. It is shown on pp. 292-293 that these "beautified" formulas are numerically equivalent versions of (4.6.8) and (4.6.9). By Proposition 4.7, the random-effects estimator (4.6.8) is consistent and asymptotically normal and the asymptotic variance is given by (4.6.9). As noted on p. 324, it should be routine to show that those conditions verified in (c) above are sufficient for the hypothesis of Proposition 4.7. In particular, the $\boldsymbol{\Sigma}_{\mathbf{xz}}$ referred to in Assumption 4.4′ can be written as $\text{E}(\widehat{\mathbf{F}}_i\otimes\mathbf{x}_i)$. In (c), we've verified that this matrix is of full column rank.

(f) Proposition 4.1, which is about the estimation of error cross moments for the multiple-equation model of Section 4.1, can easily be adapted to the common-coefficient model of Section 4.6. Besides linearity, the required assumptions are (i) that the coefficient estimate (here $\widehat{\boldsymbol{\beta}}_{\text{FE}}$) used for calculating the residual vector be consistent and (ii) that the cross moment between the vector of regressors from one equation (a row from $\widehat{\mathbf{F}}_i$) and those from another (another row from $\widehat{\mathbf{F}}_i$) exist and be finite. As seen in (d), the fixed-effects estimator $\widehat{\boldsymbol{\beta}}_{\text{FE}}$ is a GMM estimator. So it is consistent. As noted in (c), $\text{E}(\mathbf{x}_i\mathbf{x}_i')$ is non-singular. Since $\mathbf{x}_i$ contains all the elements of $\mathbf{F}_i$, the cross moment assumption is satisfied.

(g) As noted in (e), the assumptions of Proposition 4.7 holds for the present model in question. It has been verified in (f) that $\widehat{\boldsymbol{\Psi}}$ defined in (14) is consistent. Therefore, Proposition 4.7(c) holds for the present model.

(h) Since $\widehat{\boldsymbol{\eta}}_i \equiv \mathbf{C}'\boldsymbol{\eta}_i$, we have $\text{E}(\widehat{\boldsymbol{\eta}}_i\widehat{\boldsymbol{\eta}}_i') = \text{E}(\mathbf{C}'\boldsymbol{\eta}_i\boldsymbol{\eta}_i'\mathbf{C}) = \sigma_\eta^2\mathbf{C}'\mathbf{C}$ (the last equality is by (15)). By setting $\widehat{\boldsymbol{\Psi}} = \widehat{\sigma}_\eta^2\mathbf{C}'\mathbf{C}$ in the expression for $\widehat{\mathbf{W}}$ in the answer to (e) (thus setting $\widehat{\mathbf{W}} = \widehat{\sigma}_\eta^2\mathbf{C}'\mathbf{C} \otimes \left(\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i'\right)^{-1}$), the estimator can be written as a GMM estimator $(\mathbf{S}_{\mathbf{xz}}'\widehat{\mathbf{W}}\mathbf{S}_{\mathbf{xz}})^{-1}\mathbf{S}_{\mathbf{xz}}'\widehat{\mathbf{W}}\mathbf{s}_{\mathbf{xy}}$. Clearly, it is numerically equal to the GMM estimator with $\widehat{\mathbf{W}} = \mathbf{C}'\mathbf{C} \otimes \left(\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i'\right)^{-1}$, which, as was verified in (d), is the fixed-effects estimator.

(i) Evidently, replacing $\mathbf{C}$ by $\mathbf{B} \equiv \mathbf{CA}$ in (11) does not change $\mathbf{Q}$. So the fixed-effects estimator is invariant to the choice of $\mathbf{C}$. To see that the numerical values of (12) and (13) are invariant to the choice of $\mathbf{C}$, let $\check{\mathbf{F}}_i \equiv \mathbf{B}'\mathbf{F}_i$ and $\check{\mathbf{y}}_i \equiv \mathbf{B}'\mathbf{y}_i$. That is, the original M equations (5.1.1$''$) are transformed into $M-1$ equations by $\mathbf{B} = \mathbf{CA}$, not by $\mathbf{C}$. Then $\check{\mathbf{F}}_i = \mathbf{A}'\widehat{\mathbf{F}}_i$ and $\check{\mathbf{y}}_i = \mathbf{A}'\widehat{\mathbf{y}}_i$. If $\check{\boldsymbol{\Psi}}$ is the estimated error cross moment matrix when (14) is used with $\check{\mathbf{y}}_i$ replacing $\widehat{\mathbf{y}}_i$ and $\check{\mathbf{F}}_i$ replacing $\widehat{\mathbf{F}}_i$, then we have: $\check{\boldsymbol{\Psi}} = \mathbf{A}'\widehat{\boldsymbol{\Psi}}\mathbf{A}$. So

$$\check{\mathbf{F}}_i'\check{\boldsymbol{\Psi}}^{-1}\check{\mathbf{F}}_i = \widehat{\mathbf{F}}_i'\mathbf{A}(\mathbf{A}'\widehat{\boldsymbol{\Psi}}\mathbf{A})^{-1}\mathbf{A}'\widehat{\mathbf{F}}_i = \widehat{\mathbf{F}}_i'\mathbf{A}\mathbf{A}^{-1}\widehat{\boldsymbol{\Psi}}^{-1}(\mathbf{A}')^{-1}\mathbf{A}'\widehat{\mathbf{F}}_i = \widehat{\mathbf{F}}_i'\widehat{\boldsymbol{\Psi}}^{-1}\widehat{\mathbf{F}}_i.$$

Similarly, $\check{\mathbf{F}}_i'\check{\boldsymbol{\Psi}}^{-1}\check{\mathbf{y}}_i = \widehat{\mathbf{F}}_i'\widehat{\boldsymbol{\Psi}}^{-1}\widehat{\mathbf{y}}_i$.

3. From (5.1.1$''$), $\mathbf{v}_i = \mathbf{C}'(\mathbf{y}_i - \mathbf{F}_i\boldsymbol{\beta}) = \mathbf{C}'\boldsymbol{\eta}_i$. So $\text{E}(\mathbf{v}_i\mathbf{v}_i') = \text{E}(\mathbf{C}'\boldsymbol{\eta}_i\boldsymbol{\eta}_i'\mathbf{C}) = \mathbf{C}'\,\text{E}(\boldsymbol{\eta}_i\boldsymbol{\eta}_i')\mathbf{C} = \sigma_\eta^2\mathbf{C}'\mathbf{C}$. By the hint,

$$\text{plim}\ \frac{SSR}{n} = \text{trace}\left[(\mathbf{C}'\mathbf{C})^{-1}\,\sigma_\eta^2\mathbf{C}'\mathbf{C}\right] = \sigma_\eta^2\,\text{trace}[\mathbf{I}_{M-1}] = \sigma_\eta^2\cdot(M-1).$$

4. (a) $\mathbf{b}_i$ is absent from the system of M equations (or $\mathbf{b}_i$ is a zero vector).

$$\mathbf{y}_i = \begin{bmatrix} y_{i1} \\ \vdots \\ y_{iM} \end{bmatrix}, \quad \mathbf{F}_i = \begin{bmatrix} y_{i0} \\ \vdots \\ y_{i,M-1} \end{bmatrix}.$$

(b) Recursive substitution (starting with a substitution of the first equation of the system into the second) yields the equation in the hint. Multiply both sides of the equation by η_{ih} and take expectations to obtain

$$\begin{aligned}
\text{E}(y_{im}\cdot\eta_{ih}) &= \text{E}(\eta_{im}\cdot\eta_{ih}) + \rho\,\text{E}(\eta_{i,m-1}\cdot\eta_{ih}) + \cdots + \rho^{m-1}\,\text{E}(\eta_{i1}\cdot\eta_{ih}) \\
&\qquad + \frac{1-\rho^m}{1-\rho}\,\text{E}(\alpha_i\cdot\eta_{ih}) + \rho^m\,\text{E}(y_{i0}\cdot\eta_{ih}) \\
&= \text{E}(\eta_{im}\cdot\eta_{ih}) + \rho\,\text{E}(\eta_{i,m-1}\cdot\eta_{ih}) + \cdots + \rho^{m-1}\,\text{E}(\eta_{i1}\cdot\eta_{ih}) \\
&\qquad (\text{since } \text{E}(\alpha_i\cdot\eta_{ih}) = 0 \text{ and } \text{E}(y_{i0}\cdot\eta_{ih}) = 0) \\
&= \begin{cases} \rho^{m-h}\,\sigma_\eta^2 & \text{if } h = 1, 2, \ldots, m, \\ 0 & \text{if } h = m+1, m+2, \ldots. \end{cases}
\end{aligned}$$

(c) That $\mathrm{E}(y_{im} \cdot \eta_{ih}) = \rho^{m-h}\sigma_\eta^2$ for $m \geq h$ is shown in (b). Noting that $\mathbf{F}_i$ here is a vector, not a matrix, we have:

$$\begin{aligned}
\mathrm{E}(\mathbf{F}_i'\mathbf{Q}\boldsymbol{\eta}_i) &= \mathrm{E}[\mathrm{trace}(\mathbf{F}_i'\mathbf{Q}\boldsymbol{\eta}_i)] \\
&= \mathrm{E}[\mathrm{trace}(\boldsymbol{\eta}_i\mathbf{F}_i'\mathbf{Q})] \\
&= \mathrm{trace}[\mathrm{E}(\boldsymbol{\eta}_i\mathbf{F}_i')\mathbf{Q}] \\
&= \mathrm{trace}[\mathrm{E}(\boldsymbol{\eta}_i\mathbf{F}_i')(\mathbf{I}_M - \frac{1}{M}\mathbf{1}\mathbf{1}')] \\
&= \mathrm{trace}[\mathrm{E}(\boldsymbol{\eta}_i\mathbf{F}_i')] - \frac{1}{M}\,\mathrm{trace}[\mathrm{E}(\boldsymbol{\eta}_i\mathbf{F}_i')\mathbf{1}\mathbf{1}'] \\
&= \mathrm{trace}[\mathrm{E}(\boldsymbol{\eta}_i\mathbf{F}_i')] - \frac{1}{M}\mathbf{1}'\,\mathrm{E}(\boldsymbol{\eta}_i\mathbf{F}_i')\mathbf{1}.
\end{aligned}$$

By the results shown in (b), $\mathrm{E}(\boldsymbol{\eta}_i\mathbf{F}_i')$ can be written as

$$\mathrm{E}(\boldsymbol{\eta}_i\mathbf{F}_i') = \sigma_\eta^2 \begin{bmatrix} 0 & 1 & \rho & \rho^2 & \cdots & \rho^{M-2} \\ 0 & 0 & 1 & \rho & \cdots & \rho^{M-3} \\ \vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & \rho \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$

So, in the above expression for $\mathrm{E}(\mathbf{F}_i'\mathbf{Q}\boldsymbol{\eta}_i)$, $\mathrm{trace}[\mathrm{E}(\boldsymbol{\eta}_i\mathbf{F}_i')] = 0$ and

$$\begin{aligned}
\mathbf{1}'\,\mathrm{E}(\boldsymbol{\eta}_i\mathbf{F}_i')\mathbf{1} &= \text{sum of the elements of } \mathrm{E}(\boldsymbol{\eta}_i\mathbf{F}_i') \\
&= \text{sum of the first row} + \cdots + \text{sum of the last row} \\
&= \sigma_\eta^2 \left[\frac{1-\rho^{M-1}}{1-\rho} + \frac{1-\rho^{M-2}}{1-\rho} + \cdots + \frac{1-\rho}{1-\rho}\right] \\
&= \sigma_\eta^2 \frac{M - 1 - M\,\rho + \rho^M}{(1-\rho)^2}.
\end{aligned}$$

(d) (5.2.6) is violated because $\mathrm{E}(\mathbf{f}_{im} \cdot \eta_{ih}) = \mathrm{E}(y_{i,m-1} \cdot \eta_{ih}) \neq 0$ for $h \leq m-1$.

5. (a) The hint shows that

$$\mathrm{E}(\widetilde{\mathbf{F}}_i'\widetilde{\mathbf{F}}_i) = \mathrm{E}(\mathbf{Q}\mathbf{F}_i \otimes \mathbf{x}_i)'\big(\mathbf{I}_M \otimes [\mathrm{E}(\mathbf{x}_i\mathbf{x}_i')]^{-1}\big)\,\mathrm{E}(\mathbf{Q}\mathbf{F}_i \otimes \mathbf{x}_i).$$

By (5.1.15), $\mathrm{E}(\mathbf{Q}\mathbf{F}_i \otimes \mathbf{x}_i)$ is of full column rank. So the matrix product above is non-singular.

(b) By (5.1.5) and (5.1.6′), $\mathrm{E}(\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i')$ is non-singular.

(c) By the same sort of argument used in (a) and (b) and noting that $\widehat{\mathbf{F}}_i \equiv \mathbf{C}'\mathbf{F}_i$, we have

$$\mathrm{E}(\widehat{\mathbf{F}}_i'\,\boldsymbol{\Psi}^{-1}\,\widehat{\mathbf{F}}_i) = \mathrm{E}(\mathbf{C}'\mathbf{F}_i \otimes \mathbf{x}_i)'\big(\boldsymbol{\Psi}^{-1} \otimes [\mathrm{E}(\mathbf{x}_i\mathbf{x}_i')]^{-1}\big)\,\mathrm{E}(\mathbf{C}'\mathbf{F}_i \otimes \mathbf{x}_i).$$

We've verified in 2(c) that $\mathrm{E}(\mathbf{C}'\mathbf{F}_i \otimes \mathbf{x}_i)$ is of full column rank.

6. This question presumes that

$$\mathbf{x}_i = \begin{bmatrix} \mathbf{f}_{i1} \\ \vdots \\ \mathbf{f}_{iM} \\ \mathbf{b}_i \end{bmatrix} \quad \text{and} \quad \mathbf{f}_{im} = \mathbf{A}_m' \mathbf{x}_i.$$

   (a) The m-th row of $\mathbf{F}_i$ is $\mathbf{f}_{im}'$ and $\mathbf{f}_{im}' = \mathbf{x}_i'\mathbf{A}_m$.

   (b) The rank condition (5.1.15) is that $\mathrm{E}(\widetilde{\mathbf{F}}_i \otimes \mathbf{x}_i)$ be of full column rank (where $\widetilde{\mathbf{F}}_i \equiv \mathbf{QF}_i$). By the hint, $\mathrm{E}(\widetilde{\mathbf{F}}_i \otimes \mathbf{x}_i) = [\mathbf{I}_M \otimes \mathrm{E}(\mathbf{x}_i\mathbf{x}_i')](\mathbf{Q} \otimes \mathbf{I}_K)\mathbf{A}$. Since $\mathrm{E}(\mathbf{x}_i\mathbf{x}_i')$ is non-singular, $\mathbf{I}_M \otimes \mathrm{E}(\mathbf{x}_i\mathbf{x}_i')$ is non-singular. Multiplication by a non-singular matrix does not alter rank.

7. The hint is the answer.