Solution to Chapter 1 Analytical Exercises

1. (Reproducing the answer on p. 84 of the book)

$$
\begin{aligned}
(\mathbf{y}-\mathbf{X}\widetilde{\boldsymbol{\beta}})'(\mathbf{y}-\mathbf{X}\widetilde{\boldsymbol{\beta}}) &= [(\mathbf{y}-\mathbf{X}\mathbf{b})+\mathbf{X}(\mathbf{b}-\widetilde{\boldsymbol{\beta}})]'[(\mathbf{y}-\mathbf{X}\mathbf{b})+\mathbf{X}(\mathbf{b}-\widetilde{\boldsymbol{\beta}})] \\
&\qquad \text{(by the add-and-subtract strategy)} \\
&= [(\mathbf{y}-\mathbf{X}\mathbf{b})'+(\mathbf{b}-\widetilde{\boldsymbol{\beta}})'\mathbf{X}'][(\mathbf{y}-\mathbf{X}\mathbf{b})+\mathbf{X}(\mathbf{b}-\widetilde{\boldsymbol{\beta}})] \\
&= (\mathbf{y}-\mathbf{X}\mathbf{b})'(\mathbf{y}-\mathbf{X}\mathbf{b})+(\mathbf{b}-\widetilde{\boldsymbol{\beta}})'\mathbf{X}'(\mathbf{y}-\mathbf{X}\mathbf{b}) \\
&\qquad +(\mathbf{y}-\mathbf{X}\mathbf{b})'\mathbf{X}(\mathbf{b}-\widetilde{\boldsymbol{\beta}})+(\mathbf{b}-\widetilde{\boldsymbol{\beta}})'\mathbf{X}'\mathbf{X}(\mathbf{b}-\widetilde{\boldsymbol{\beta}}) \\
&= (\mathbf{y}-\mathbf{X}\mathbf{b})'(\mathbf{y}-\mathbf{X}\mathbf{b})+2(\mathbf{b}-\widetilde{\boldsymbol{\beta}})'\mathbf{X}'(\mathbf{y}-\mathbf{X}\mathbf{b})+(\mathbf{b}-\widetilde{\boldsymbol{\beta}})'\mathbf{X}'\mathbf{X}(\mathbf{b}-\widetilde{\boldsymbol{\beta}}) \\
&\qquad \text{(since } (\mathbf{b}-\widetilde{\boldsymbol{\beta}})'\mathbf{X}'(\mathbf{y}-\mathbf{X}\mathbf{b})=(\mathbf{y}-\mathbf{X}\mathbf{b})'\mathbf{X}(\mathbf{b}-\widetilde{\boldsymbol{\beta}})\text{)} \\
&= (\mathbf{y}-\mathbf{X}\mathbf{b})'(\mathbf{y}-\mathbf{X}\mathbf{b})+(\mathbf{b}-\widetilde{\boldsymbol{\beta}})'\mathbf{X}'\mathbf{X}(\mathbf{b}-\widetilde{\boldsymbol{\beta}}) \\
&\qquad \text{(since } \mathbf{X}'(\mathbf{y}-\mathbf{X}\mathbf{b})=\mathbf{0} \text{ by the normal equations)} \\
&\geq (\mathbf{y}-\mathbf{X}\mathbf{b})'(\mathbf{y}-\mathbf{X}\mathbf{b}) \\
&\qquad \text{(since } (\mathbf{b}-\widetilde{\boldsymbol{\beta}})'\mathbf{X}'\mathbf{X}(\mathbf{b}-\widetilde{\boldsymbol{\beta}})=\mathbf{z}'\mathbf{z}=\sum_{i=1}^{n} z_i^2 \geq 0 \text{ where } \mathbf{z}\equiv\mathbf{X}(\mathbf{b}-\widetilde{\boldsymbol{\beta}})\text{)}.
\end{aligned}
$$

2. (a), (b). If $\mathbf{X}$ is an $n \times K$ matrix of full column rank, then $\mathbf{X}'\mathbf{X}$ is symmetric and invertible. It is very straightforward to show (and indeed you've been asked to show in the text) that $\mathbf{M_X} \equiv \mathbf{I}_n - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ is symmetric and idempotent and that $\mathbf{M_X X} = \mathbf{0}$. In this question, set $\mathbf{X} = \mathbf{1}$ (vector of ones).

   (c)

$$
\begin{aligned}
\mathbf{M_1 y} &= [\mathbf{I}_n - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}']\mathbf{y} \\
&= \mathbf{y} - \frac{1}{n}\mathbf{1}\mathbf{1}'\mathbf{y} \qquad \text{(since } \mathbf{1}'\mathbf{1} = n\text{)} \\
&= \mathbf{y} - \frac{1}{n}\mathbf{1}\sum_{i=1}^{n} y_i = \mathbf{y} - \mathbf{1}\cdot\overline{y}
\end{aligned}
$$

   (d) Replace "$\mathbf{y}$" by "$\mathbf{X}$" in (c).

3. Special case of the solution to the next exercise.

4. From the normal equations (1.2.3) of the text, we obtain

   (a)

$$
\begin{bmatrix} \mathbf{X}_1' \\ \mathbf{X}_2' \end{bmatrix} [\mathbf{X}_1 \vdots \mathbf{X}_2] \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1' \\ \mathbf{X}_2' \end{bmatrix} \mathbf{y}.
$$

   Using the rules of multiplication of partitioned matrices, it is straightforward to derive (∗) and (∗∗) from the above.

(b) By premultiplying both sides of $(*)$ in the question by $\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}$, we obtain

$$\begin{aligned}\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_1\mathbf{b}_1 &= -\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2\mathbf{b}_2 + \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{y} \\ \Leftrightarrow \quad \mathbf{X}_1\mathbf{b}_1 &= -\mathbf{P}_1\mathbf{X}_2\mathbf{b}_2 + \mathbf{P}_1\mathbf{y}\end{aligned}$$

Substitution of this into $(**)$ yields

$$\begin{aligned}\mathbf{X}_2'(-\mathbf{P}_1\mathbf{X}_2\mathbf{b}_2 + \mathbf{P}_1\mathbf{y}) + \mathbf{X}_2'\mathbf{X}_2\mathbf{b}_2 &= \mathbf{X}_2'\mathbf{y} \\ \Leftrightarrow \quad \mathbf{X}_2'(\mathbf{I} - \mathbf{P}_1)\mathbf{X}_2\mathbf{b}_2 &= \mathbf{X}_2'(\mathbf{I} - \mathbf{P}_1)\mathbf{y} \\ \Leftrightarrow \quad \mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2\mathbf{b}_2 &= \mathbf{X}_2'\mathbf{M}_1\mathbf{y} \\ \Leftrightarrow \quad \mathbf{X}_2'\mathbf{M}_1'\mathbf{M}_1\mathbf{X}_2\mathbf{b}_2 &= \mathbf{X}_2'\mathbf{M}_1'\mathbf{M}_1\mathbf{y} \quad \text{(since } \mathbf{M}_1 \text{ is symmetric \& idempotent)} \\ \Leftrightarrow \quad \widetilde{\mathbf{X}}_2'\widetilde{\mathbf{X}}_2\mathbf{b}_2 &= \widetilde{\mathbf{X}}_2'\widetilde{\mathbf{y}}.\end{aligned}$$

Therefore,

$$\mathbf{b}_2 = (\widetilde{\mathbf{X}}_2'\widetilde{\mathbf{X}}_2)^{-1}\widetilde{\mathbf{X}}_2'\widetilde{\mathbf{y}}$$

(The matrix $\widetilde{\mathbf{X}}_2'\widetilde{\mathbf{X}}_2$ is invertible because $\widetilde{\mathbf{X}}_2$ is of full column rank. To see that $\widetilde{\mathbf{X}}_2$ is of full column rank, suppose not. Then there exists a non-zero vector $\mathbf{c}$ such that $\widetilde{\mathbf{X}}_2\mathbf{c} = \mathbf{0}$. But $\widetilde{\mathbf{X}}_2\mathbf{c} = \mathbf{X}_2\mathbf{c} - \mathbf{X}_1\mathbf{d}$ where $\mathbf{d} \equiv (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2\mathbf{c}$. That is, $\mathbf{X}\boldsymbol{\pi} = \mathbf{0}$ for $\boldsymbol{\pi} \equiv \begin{bmatrix}-\mathbf{d} \\ \mathbf{c}\end{bmatrix}$. This is a contradiction because $\mathbf{X} = [\mathbf{X}_1 \vdots \mathbf{X}_2]$ is of full column rank and $\boldsymbol{\pi} \neq \mathbf{0}$.)

(c) By premultiplying both sides of $\mathbf{y} = \mathbf{X}_1\mathbf{b}_1 + \mathbf{X}_2\mathbf{b}_2 + \mathbf{e}$ by $\mathbf{M}_1$, we obtain

$$\mathbf{M}_1\mathbf{y} = \mathbf{M}_1\mathbf{X}_1\mathbf{b}_1 + \mathbf{M}_1\mathbf{X}_2\mathbf{b}_2 + \mathbf{M}_1\mathbf{e}.$$

Since $\mathbf{M}_1\mathbf{X}_1 = \mathbf{0}$ and $\widetilde{\mathbf{y}} \equiv \mathbf{M}_1\mathbf{y}$, the above equation can be rewritten as

$$\begin{aligned}\widetilde{\mathbf{y}} &= \mathbf{M}_1\mathbf{X}_2\mathbf{b}_2 + \mathbf{M}_1\mathbf{e} \\ &= \widetilde{\mathbf{X}}_2\mathbf{b}_2 + \mathbf{M}_1\mathbf{e}.\end{aligned}$$

$\mathbf{M}_1\mathbf{e} = \mathbf{e}$ because

$$\begin{aligned}\mathbf{M}_1\mathbf{e} &= (\mathbf{I} - \mathbf{P}_1)\mathbf{e} \\ &= \mathbf{e} - \mathbf{P}_1\mathbf{e} \\ &= \mathbf{e} - \mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{e} \\ &= \mathbf{e} \qquad \text{(since } \mathbf{X}_1'\mathbf{e} = \mathbf{0} \text{ by normal equations).}\end{aligned}$$

(d) From (b), we have

$$\begin{aligned}\mathbf{b}_2 &= (\widetilde{\mathbf{X}}_2'\widetilde{\mathbf{X}}_2)^{-1}\widetilde{\mathbf{X}}_2'\widetilde{\mathbf{y}} \\ &= (\widetilde{\mathbf{X}}_2'\widetilde{\mathbf{X}}_2)^{-1}\mathbf{X}_2'\mathbf{M}_1'\mathbf{M}_1\mathbf{y} \\ &= (\widetilde{\mathbf{X}}_2'\widetilde{\mathbf{X}}_2)^{-1}\widetilde{\mathbf{X}}_2'\mathbf{y}.\end{aligned}$$

Therefore, $\mathbf{b}_2$ is the OLS coefficient estimator for the regression $\mathbf{y}$ on $\widetilde{\mathbf{X}}_2$. The residual vector from the regression is

$$\begin{aligned}\mathbf{y} - \widetilde{\mathbf{X}}_2\mathbf{b}_2 &= (\mathbf{y} - \widetilde{\mathbf{y}}) + (\widetilde{\mathbf{y}} - \widetilde{\mathbf{X}}_2\mathbf{b}_2) \\ &= (\mathbf{y} - \mathbf{M}_1\mathbf{y}) + (\widetilde{\mathbf{y}} - \widetilde{\mathbf{X}}_2\mathbf{b}_2) \\ &= (\mathbf{y} - \mathbf{M}_1\mathbf{y}) + \mathbf{e} \qquad \text{(by (c))} \\ &= \mathbf{P}_1\mathbf{y} + \mathbf{e}.\end{aligned}$$

This does not equal $\mathbf{e}$ because $\mathbf{P}_1\mathbf{y}$ is not necessarily zero. The *SSR* from the regression of $\mathbf{y}$ on $\widetilde{\mathbf{X}}_2$ can be written as

$$\begin{aligned}(\mathbf{y}-\widetilde{\mathbf{X}}_2\mathbf{b}_2)'(\mathbf{y}-\widetilde{\mathbf{X}}_2\mathbf{b}_2) &= (\mathbf{P}_1\mathbf{y}+\mathbf{e})'(\mathbf{P}_1\mathbf{y}+\mathbf{e}) \\ &= (\mathbf{P}_1\mathbf{y})'(\mathbf{P}_1\mathbf{y})+\mathbf{e}'\mathbf{e} \qquad (\text{since } \mathbf{P}_1\mathbf{e}=\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{e}=\mathbf{0}).\end{aligned}$$

This does not equal $\mathbf{e}'\mathbf{e}$ if $\mathbf{P}_1\mathbf{y}$ is not zero.

(e) From (c), $\widetilde{\mathbf{y}}=\widetilde{\mathbf{X}}_2\mathbf{b}_2+\mathbf{e}$. So

$$\begin{aligned}\widetilde{\mathbf{y}}'\widetilde{\mathbf{y}} &= (\widetilde{\mathbf{X}}_2\mathbf{b}_2+\mathbf{e})'(\widetilde{\mathbf{X}}_2\mathbf{b}_2+\mathbf{e}) \\ &= \mathbf{b}_2'\widetilde{\mathbf{X}}_2'\widetilde{\mathbf{X}}_2\mathbf{b}_2+\mathbf{e}'\mathbf{e} \qquad (\text{since } \widetilde{\mathbf{X}}_2\mathbf{e}=\mathbf{0}).\end{aligned}$$

Since $\mathbf{b}_2=(\widetilde{\mathbf{X}}_2'\widetilde{\mathbf{X}}_2)^{-1}\widetilde{\mathbf{X}}_2'\mathbf{y}$, we have $\mathbf{b}_2'\widetilde{\mathbf{X}}_2'\widetilde{\mathbf{X}}_2\mathbf{b}_2=\widetilde{\mathbf{y}}'\mathbf{X}_2(\mathbf{X}_2'\mathbf{M}_1\mathbf{X}_2)^{-1}\mathbf{X}_2\widetilde{\mathbf{y}}$.

(f) (i) Let $\widehat{\mathbf{b}}_1$ be the OLS coefficient estimator for the regression of $\widetilde{\mathbf{y}}$ on $\mathbf{X}_1$. Then

$$\begin{aligned}\widehat{\mathbf{b}}_1 &= (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\widetilde{\mathbf{y}} \\ &= (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{M}_1\mathbf{y} \\ &= (\mathbf{X}_1'\mathbf{X}_1)^{-1}(\mathbf{M}_1\mathbf{X}_1)'\mathbf{y} \\ &= \mathbf{0} \qquad (\text{since } \mathbf{M}_1\mathbf{X}_1=\mathbf{0}).\end{aligned}$$

So $SSR_1=(\widetilde{\mathbf{y}}-\mathbf{X}_1\widehat{\mathbf{b}}_1)'(\widetilde{\mathbf{y}}-\mathbf{X}_1\widehat{\mathbf{b}}_1)=\widetilde{\mathbf{y}}'\widetilde{\mathbf{y}}$.

(ii) Since the residual vector from the regression of $\widetilde{\mathbf{y}}$ on $\widetilde{\mathbf{X}}_2$ equals $\mathbf{e}$ by (c), $SSR_2=\mathbf{e}'\mathbf{e}$.

(iii) From the Frisch-Waugh Theorem, the residuals from the regression of $\widetilde{\mathbf{y}}$ on $\mathbf{X}_1$ and $\mathbf{X}_2$ equal those from the regression of $\mathbf{M}_1\widetilde{\mathbf{y}}$ $(=\widetilde{\mathbf{y}})$ on $\mathbf{M}_1\mathbf{X}_2$ $(=\widetilde{\mathbf{X}}_2)$. So $SSR_3=\mathbf{e}'\mathbf{e}$.

5. (a) The hint is as good as the answer.

(b) Let $\widehat{\boldsymbol{\varepsilon}}\equiv\mathbf{y}-\mathbf{X}\widehat{\boldsymbol{\beta}}$, the residuals from the restricted regression. By using the add-and-subtract strategy, we obtain

$$\widehat{\boldsymbol{\varepsilon}}\equiv\mathbf{y}-\mathbf{X}\widehat{\boldsymbol{\beta}}=(\mathbf{y}-\mathbf{X}\mathbf{b})+\mathbf{X}(\mathbf{b}-\widehat{\boldsymbol{\beta}}).$$

So

$$\begin{aligned}SSR_R &= [(\mathbf{y}-\mathbf{X}\mathbf{b})+\mathbf{X}(\mathbf{b}-\widehat{\boldsymbol{\beta}})]'[(\mathbf{y}-\mathbf{X}\mathbf{b})+\mathbf{X}(\mathbf{b}-\widehat{\boldsymbol{\beta}})] \\ &= (\mathbf{y}-\mathbf{X}\mathbf{b})'(\mathbf{y}-\mathbf{X}\mathbf{b})+(\mathbf{b}-\widehat{\boldsymbol{\beta}})'\mathbf{X}'\mathbf{X}(\mathbf{b}-\widehat{\boldsymbol{\beta}}) \qquad (\text{since } \mathbf{X}'(\mathbf{y}-\mathbf{X}\mathbf{b})=\mathbf{0}).\end{aligned}$$

But $SSR_U=(\mathbf{y}-\mathbf{X}\mathbf{b})'(\mathbf{y}-\mathbf{X}\mathbf{b})$, so

$$\begin{aligned}SSR_R-SSR_U &= (\mathbf{b}-\widehat{\boldsymbol{\beta}})'\mathbf{X}'\mathbf{X}(\mathbf{b}-\widehat{\boldsymbol{\beta}}) \\ &= (\mathbf{R}\mathbf{b}-\mathbf{r})'[\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}']^{-1}(\mathbf{R}\mathbf{b}-\mathbf{r}) \qquad (\text{using the expresion for } \widehat{\boldsymbol{\beta}} \text{ from (a)}) \\ &= \boldsymbol{\lambda}'\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}'\boldsymbol{\lambda} \qquad (\text{using the expresion for } \boldsymbol{\lambda} \text{ from (a)}) \\ &= \widehat{\boldsymbol{\varepsilon}}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\widehat{\boldsymbol{\varepsilon}} \qquad (\text{by the first order conditions that } \mathbf{X}'(\mathbf{y}-\mathbf{X}\widehat{\boldsymbol{\beta}})=\mathbf{R}'\boldsymbol{\lambda}) \\ &= \widehat{\boldsymbol{\varepsilon}}'\mathbf{P}\widehat{\boldsymbol{\varepsilon}}.\end{aligned}$$

(c) The F-ratio is defined as

$$F\equiv\frac{(\mathbf{R}\mathbf{b}-\mathbf{r})'[\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}']^{-1}(\mathbf{R}\mathbf{b}-\mathbf{r})/r}{s^2} \qquad (\text{where } r=\#\mathbf{r}) \tag{1.4.9}$$

Since $(\mathbf{Rb}-\mathbf{r})'[\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}']^{-1}(\mathbf{Rb}-\mathbf{r}) = SSR_R - SSR_U$ as shown above, the F-ratio can be rewritten as

$$\begin{aligned} F &= \frac{(SSR_R - SSR_U)/r}{s^2} \\ &= \frac{(SSR_R - SSR_U)/r}{\mathbf{e}'\mathbf{e}/(n-K)} \\ &= \frac{(SSR_R - SSR_U)/r}{SSR_U/(n-K)} \end{aligned}$$

Therefore, (1.4.9)=(1.4.11).

6. (a) **Unrestricted model**: $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where

$$\underset{(N\times 1)}{\mathbf{y}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \underset{(N\times K)}{\mathbf{X}} = \begin{bmatrix} 1 & x_{12} & \dots & x_{1K} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n2} & \dots & x_{nK} \end{bmatrix}, \quad \underset{(K\times 1)}{\boldsymbol{\beta}} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}.$$

Restricted model: $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, $\mathbf{R}\boldsymbol{\beta} = \mathbf{r}$, where

$$\underset{((K-1)\times K)}{\mathbf{R}} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & 1 \end{bmatrix}, \quad \underset{((K-1)\times 1)}{\mathbf{r}} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Obviously, the restricted OLS estimator of $\boldsymbol{\beta}$ is

$$\underset{(K\times 1)}{\widehat{\boldsymbol{\beta}}} = \begin{bmatrix} \overline{y} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad \text{So } \mathbf{X}\widehat{\boldsymbol{\beta}} = \begin{bmatrix} \overline{y} \\ \overline{y} \\ \vdots \\ \overline{y} \end{bmatrix} = \mathbf{1}\cdot \overline{y}.$$

(You can use the formula for the unrestricted OLS derived in the previous exercise, $\widehat{\boldsymbol{\beta}} = \mathbf{b} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}'[\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}']^{-1}(\mathbf{Rb}-\mathbf{r})$, to verify this.) If SSR_U and SSR_R are the minimized sums of squared residuals from the unrestricted and restricted models, they are calculated as

$$\begin{aligned} SSR_R &= (\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}})'(\mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}) = \sum_{i=1}^{n}(y_i - \overline{y})^2 \\ SSR_U &= (\mathbf{y} - \mathbf{Xb})'(\mathbf{y} - \mathbf{Xb}) = \mathbf{e}'\mathbf{e} = \sum_{i=1}^{n} e_i^2 \end{aligned}$$

Therefore,

$$SSR_R - SSR_U = \sum_{i=1}^{n}(y_i - \overline{y})^2 - \sum_{i=1}^{n} e_i^2. \tag{A}$$

On the other hand,

$$\begin{aligned}(\mathbf{b}-\widehat{\boldsymbol{\beta}})'(\mathbf{X}'\mathbf{X})(\mathbf{b}-\widehat{\boldsymbol{\beta}}) &= (\mathbf{X}\mathbf{b}-\mathbf{X}\widehat{\boldsymbol{\beta}})'(\mathbf{X}\mathbf{b}-\mathbf{X}\widehat{\boldsymbol{\beta}}) \\ &= \sum_{i=1}^{n}(\widehat{y}_i-\overline{y})^2.\end{aligned}$$

Since $SSR_R - SSR_U = (\mathbf{b}-\widehat{\boldsymbol{\beta}})'(\mathbf{X}'\mathbf{X})(\mathbf{b}-\widehat{\boldsymbol{\beta}})$ (as shown in Exercise 5(b)),

$$\sum_{i=1}^{n}(y_i-\overline{y})^2 - \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n}(\widehat{y}_i-\overline{y})^2. \tag{B}$$

(b)

$$\begin{aligned}F &= \frac{(SSR_R - SSR_U)/(K-1)}{\sum_{i=1}^n e_i^2/(n-K)} \qquad \text{(by Exercise 5(c))} \\ &= \frac{(\sum_{i=1}^n (y_i-\overline{y})^2 - \sum_{i=1}^n e_i^2)/(K-1)}{\sum_{i=1}^n e_i^2/(n-K)} \qquad \text{(by equation (A) above)} \\ &= \frac{\sum_{i=1}^n (\widehat{y}_i-\overline{y})^2/(K-1)}{\sum_{i=1}^n e_i^2/(n-K)} \qquad \text{(by equation (B) above)} \\ &= \frac{\frac{\sum_{i=1}^n (\widehat{y}_i-\overline{y})^2/(K-1)}{\sum_{i=1}^n (y_i-\overline{y})^2}}{\frac{\sum_{i=1}^n e_i^2/(n-K)}{\sum_{i=1}^n (y_i-\overline{y})^2}} \qquad \text{(by dividing both numerator \& denominator by } \sum_{i=1}^n (y_i-\overline{y})^2) \\ &= \frac{R^2/(K-1)}{(1-R^2)/(n-K)} \qquad \text{(by the definition or } R^2).\end{aligned}$$

7. (Reproducing the answer on pp. 84-85 of the book)

(a) $\widehat{\boldsymbol{\beta}}_{\text{GLS}} - \boldsymbol{\beta} = \mathbf{A}\boldsymbol{\varepsilon}$ where $\mathbf{A} \equiv (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}$ and $\mathbf{b} - \widehat{\boldsymbol{\beta}}_{\text{GLS}} = \mathbf{B}\boldsymbol{\varepsilon}$ where $\mathbf{B} \equiv (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' - (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}$. So

$$\begin{aligned}&\text{Cov}(\widehat{\boldsymbol{\beta}}_{\text{GLS}} - \boldsymbol{\beta}, \mathbf{b} - \widehat{\boldsymbol{\beta}}_{\text{GLS}}) \\ &= \text{Cov}(\mathbf{A}\boldsymbol{\varepsilon}, \mathbf{B}\boldsymbol{\varepsilon}) \\ &= \mathbf{A}\,\text{Var}(\boldsymbol{\varepsilon})\mathbf{B}' \\ &= \sigma^2\mathbf{A}\mathbf{V}\mathbf{B}'.\end{aligned}$$

It is straightforward to show that $\mathbf{A}\mathbf{V}\mathbf{B}' = \mathbf{0}$.

(b) For the choice of $\mathbf{H}$ indicated in the hint,

$$\text{Var}(\widehat{\boldsymbol{\beta}}) - \text{Var}(\widehat{\boldsymbol{\beta}}_{\text{GLS}}) = -\mathbf{C}\mathbf{V}_q^{-1}\mathbf{C}'.$$

If $\mathbf{C} \neq \mathbf{0}$, then there exists a nonzero vector $\mathbf{z}$ such that $\mathbf{C}'\mathbf{z} \equiv \mathbf{v} \neq \mathbf{0}$. For such $\mathbf{z}$,

$$\mathbf{z}'[\text{Var}(\widehat{\boldsymbol{\beta}}) - \text{Var}(\widehat{\boldsymbol{\beta}}_{\text{GLS}})]\mathbf{z} = -\mathbf{v}'\mathbf{V}_q^{-1}\mathbf{v} < 0 \quad \text{(since } \mathbf{V}_q \text{ is positive definite)},$$

which is a contradiction because $\widehat{\boldsymbol{\beta}}_{\text{GLS}}$ is efficient.